Chapter 1 : Matrix calculus

In linear algebra, the set of matrices is denoted by $\mathcal{M}_{m\times n}(\mathbb{R})$ for real matrices (real entries) and $\mathcal{M}_{m\times n}(\mathbb{C})$ for complex matrices, where m and n represent the number of rows and columns, respectively. The rows and columns of a matrix generate vector spaces called the row space and the column space.

1 Definition

There is two representation of a vector :

1. Horizontal array, also called row vector

$$V^* = (b_1, b_2, ..., b_m)$$

   where m is the dimension of this row vector. $b_1, b_2, ..., b_m$ are the components (or elements) and m represents the number of these components.

2. Vertical array, also called column vector :

$$V = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix is a rectangular representation of numbers or functions :

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

Where n is the number of columns and m is the number of rows. The size or the number of elements of a matrix is given by $m \times n$. In the case of a square matrix where rows number=columns number= n, the size is just $n \times n$. Based on this definition, one can conclude that :

- Row vector is just a $1 \times n$ matrix
- Column vector is a $n \times 1$ matrix.

Example 1.

Let $A_{2\times 3} = \begin{pmatrix} 2 & -5 & 7 \\ 3 & 0 & 1 \end{pmatrix}$. The number of elements (size) of A is $2 \times 3 = 6$ elements. Where $a_{11} = 2$,$a_{12} = -5$,$a_{13} = 7$, $a_{21} = 3$, $a_{22} = 0$ and $a_{23} = 1$.

Let B be a matrix defined by $b_{ij} = |a_{ij}| + 2\,(a_{ij} - 3)$, then

$$B = \begin{pmatrix} 0 & -11 & 15 \\ 3 & -6 & -3 \end{pmatrix}$$

2 Operations on matices

2.1 Addition :

Let A, B and C be a three $n \times n$ matrices. The matrix $C = A + B$ can be given by $c_{ij} = a_{ij} + b_{ij}$ and each element is calculated as :

$$\begin{aligned} c_{11} &= a_{11} + b_{11} \\ c_{12} &= a_{12} + b_{12} \\ &\vdots \\ &\vdots \\ c_{nn} &= a_{nn} + b_{nn} \end{aligned}$$

Where A and B are of the same number of rows and columns.

If $C = A + B$, then $B = C - A$ and $A = C - B$.

Example

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 6 \\ 1 & -2 \end{pmatrix},\ B = \begin{pmatrix} 1 & -5 \\ -2 & 3 \\ 4 & 7 \end{pmatrix}\ C = A + B = \begin{pmatrix} 2+1 & 4+(-5) \\ 1+(-2) & 6+3 \\ 1+4 & -2+7 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & -1 \\ -1 & 9 \\ 5 & 5 \end{pmatrix}$$

2.2 Multiplication

Multiplication by a scalair

Let λ be a real number, then the multiplication of λ by a matrix A is given by :

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1\ell} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{m\ell} \end{pmatrix}$$

Column and row multiple of a matrix

Let $V^* = (b_1, b_2, ..., b_m)$ and $V = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ and let A be a matrix of $m \times m$.

$V^*A = (b_1a_{11} + b_2a_{21} + .. + b_ma_{m1}, ..., ...b_1a_{1m} + b_2a_{2m} + a_{mm}b_m)$

$$AV = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1m}bm \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2m}bm \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mm}bm \end{pmatrix}$$

Matrices multiplication

Let A be a matrix of $m \times \ell$, and B a matrix of $\ell \times n$. If $C = A_{m\times\ell}B_{\ell\times n}$, then C is a matrix of $m \times n$ calculated as follows :

$$C = AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\ell} \\ a_{21} & a_{22} & \cdots & a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m\ell} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\ell 1} & B_{\ell 2} & \cdots & B_{\ell n} \end{pmatrix}$$

$$\begin{gathered} c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1\ell}b_{\ell 1} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1\ell}b_{\ell 2} \\ \vdots \\ c_{1n} = a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1\ell}b_{\ell n} \\ c_{21} = a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2\ell}b_{\ell 2} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2\ell}b_{\ell 2} \\ \vdots \\ c_{2n} = a_{21}b_{1n} + a_{22}b_{2n} + \cdots + a_{2\ell}b_{\ell n} \\ \vdots \\ c_{m1} = a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{m\ell}b_{\ell 1} \\ c_{m2} = a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{m\ell}b_{\ell 2} \\ \vdots \\ c_{mn} = a_{m1}b_{1n} + a_{m2}b_{2n} + \cdots + a_{m\ell}b_{\ell n} \end{gathered}$$

In general, the entries of the product of two matrices are defined by

$$c_{ij} = \sum_{k=1}^{\ell} a_{ik}b_{kj}$$

In simpler terms, matrix multiplication is performed as follows : to compute the entry in position (i, j), one takes the i-th row of the first matrix and multiplies it by the j-th column of the second matrix, summing the resulting products.

Example :

Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ -2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 & -3 \\ 4 & 1 & -2 & 3 \end{pmatrix}$:

$$AB = \begin{pmatrix} 2-4 & 4-1 & -2+2 & -6-3 \\ 1+12 & 2+3 & -1-6 & -3+9 \\ -2-12 & -4-3 & 2+6 & 6-9 \end{pmatrix}$$

$$AB = \begin{pmatrix} -2 & 3 & 0 & -9 \\ 13 & 5 & -7 & 6 \\ -14 & -7 & 8 & -3 \end{pmatrix}$$

Properties :

Let $A,\ B,\ C,$ be three $m \times n$ matrices and D be a $n \times p$ matrices and let $\lambda_1$and $\lambda_2$be two real numbers :

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $1A = A$ and $0A = 0_{m\times n}$
4. $\lambda_1(A + B) = \lambda_1 A + \lambda_1 B$
5. $\lambda_1(\lambda_2 A) = \lambda_2(\lambda_1 A) = (\lambda_1\lambda_2)A$
6. $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$
7. $(A + B)D = AD + BD$
8. $(\lambda_1 A)D = \lambda_1(AD) = A(\lambda_1 D)$
9. if A is a linear combination of B and C, then $A = \lambda_1 B + \lambda_2 C$.

3 Special Matrices

Square matrix

A matrix is square if number of rows=number of columns

Diagonal matrix

A matrix A is diagonal, if it is a square matrix, and all its elements are zero except for those on the main diagonal. In other words, the off-diagonal elements are zero.

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

Identity matrix

The identity matrix denoted by I is a diagonal matrix and all of its diagonal elements are equal to 1.

$$I_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With the following properties : $IA = AI = A$, where A is a $(n \times n)$ matrix.

Symmetric matrix

A matrix A is said to be symmetric if :

- A is a square matrix
- $a_{ij} = a_{ji}$

As an example : $A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{pmatrix}$

Anti-symmetric matrix

A matrix A is said to be antisymmetric if :

- A is square matrix
- $a_{ij} = -a_{ji}$

The diagonal elements of an antisymmetric matrix are zero, since $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.

As an example : $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$

Triangular matrix :

A triangular matrix is a matrix all of its below (or above) the main diagonal are zero.

Example :

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 15 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -1 & 10 & 6 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & 5 & 11 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

4 Matrix of a linear transformations

Let $T : V \longrightarrow U$ be a linear transformation where V and U are two vector spaces with dimensions n and m respectively. The linear transformation T is written as :

$$T(u) = Au,$$

where u is a $1 \times m$ vector, A is a $(m \times n)$ matrix and Au is a $1 \times n$ vector.

- Let $\{e_1, e_2, ..., e_n\}$be the basis of V and $\{f_1, f_2, ..., f_m\}$be the basis of U then

$$T(e_j) = \sum_{i=1}^{m} a_{ij} f_i,$$

,

where $1 \leq j \leq n$. The matrix $A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called the matrix representation of T.

Example :

Consider the linear transformation

$$\begin{aligned} T : R^2 &\longrightarrow R^3 \\ (x, y) &\longrightarrow (x + y, 2x, x - 2y) \end{aligned}$$

from this we get $m = 3$ and $n = 2$ then the matrix representation of T is a 3×2 matrix determined as follows :

$$\begin{aligned} T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} &= \sum_{i=1}^{3} a_{i1} f_i = a_{11} f_1 + a_{21} f_2 + a_{31} f_3 \\ &= a_{11} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_{31} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} &= \sum_{i=1}^{3} a_{i2} f_i = a_{12} f_1 + a_{22} f_2 + a_{32} f_3 \end{aligned}$$

Solve this system to get : $a_{11} = 1, a_{12} = 1, a_{21} = 2, a_{22} = 0, a_{31} = 1, a_{32} = -2$. The matrix representation of T it can be given by the two row vectors $T(e_1)$ and $T(e_2)$ as

$$A = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -2 \end{pmatrix}$$

Special matrix of linear transformation

The following table gives some (2×2) matrice of linear transformations

Linear transformation	Matrix representation
The zero linear map	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Scaling matrix	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
Rotation Matrix : rotates a vector by an angle θ	$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
Rotation by $\pi/2$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Reflection in line	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection in origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Properties :

- T is invertible (or bijective) if its matrix representation A nonsingular ($det(A) \neq 0$).
- If T is bijective, then the matrix inverse $A^{-1}$of A is the matrix representation of T^{-1} and $dim(U) = dim(V)$.

Trace of a matrix

Trace of a matrix A denoted by $tr(A)$ is the sum of diagonal elements and it is defined only for sequare matrices.

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Transpose of a Matrix

The transpose of a matrix $A_{m\times n}$ denoted by A^T is a $(n \times m)$ matrix given by :

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

This means the rows of A become columns in A^T.

Example

The transpose of $v = (1, 2, 3)$ is $v^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. The transpose of $A = \begin{pmatrix} 1 & 3 \\ -1 & -2 \\ 2 & 5 \end{pmatrix}$ is $A^T = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -2 & 5 \end{pmatrix}$.

Properties

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$

5 Determinants

The determinat of a matrix is a number and it is defined only for square matrices. The determinant of a matrice $A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is given by :

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The determinant of $A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is the number

$$\begin{aligned} det(A) &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11} \left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12} \left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13} \left(a_{21}a_{32} - a_{31}a_{22}\right) \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \end{aligned}$$

where $C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $C_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and $C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

In general, the determinant of a matrix $A_{n\times n}$ is defined for each $1 \leq i \leq n$ by :

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

where $C_{ij} = (-1)^{i+j} M_{ij}$ are the cofactor expansion of $det(A)$ defined as

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Here M_{ij} is the determinant of the submatrix obtained by deleting the i-th row and the j-th column of A.

Example :

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & -1 \\ -2 & 2 & 1 \end{pmatrix}$.

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} = (-1)^2 \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} = 4 + 2 = 6$$
$$C_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = -(1-2) = 1$$
$$C_{13} = (-1)^{1+3}M_{13} = (-1)^4 \begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} = 2 + 8 = 10$$

Then we get $det(A) = 6a_{11} + a_{12} + 10a_{13} = 6 + 2 + 30 = 38$. Also, we can verify that

$$\begin{aligned} det(A) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{aligned}.$$

In the case of a (4×4) matrices the determinant is given by :

$$det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$
$$+a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Properties :

- $det(AB) = det(A)det(B)$
- $det(A^t) = det(A)$
- Determinant of a matrix with two identical rows is zero.
- Determinant of a matrix with zero row is zero.
- If A is a $(n \times n)$ triangular matrix, $det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$.
- If a matrix A' is just a swap of two rows or columns of A, then $det(A') = -det(A)$.
- If a matrix A' is just a multiple of a row or column of A by scalar k, then $det(A') = kdet(A)$.
- Determinant remain unchanged under rows manipilations have the form $R_i \longrightarrow R_i + kR_j$.

Row reduction :

Row reduction is a method to calculate the determinants. In this method, the matrix is reduced to be triangular. This is done based on the properties of determianats mentioned above.

Example :

Let $A = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix}$. The row reduction methode is used as follows :

$\begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix} \xrightarrow{R_3 \longrightarrow R_3 - R_1} \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ -1 & -2 & 2 \end{pmatrix} \xrightarrow{R_2 \longrightarrow R_2 + 3R_1} \begin{pmatrix} 5 & 1 & 2 \\ 0 & -6 & 13 \\ -1 & -2 & 2 \end{pmatrix}$

swap of two rows $det(A) \longrightarrow -det(A)$

$\begin{pmatrix} 5 & 1 & 2 \\ 0 & -6 & 13 \\ -1 & -2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -2 & 2 \\ 0 & -6 & 13 \\ 5 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 \longrightarrow R_3 + 5R_1} \begin{pmatrix} -1 & -2 & 2 \\ 0 & -6 & 13 \\ 0 & -9 & 12 \end{pmatrix}$

$\xrightarrow{R_3 \longrightarrow R_3 - \frac{3}{2} R_2} \begin{pmatrix} -1 & -2 & 2 \\ 0 & -6 & 13 \\ 0 & 0 & 12 - \frac{3}{2}(13) \end{pmatrix}$

$det(A) = -(-1)(-6)(-7.5) = 45$

6 Inverse matrix

A matrix $A_{n\times n}$ is said to be invertible or nonsingular if there is a matrix $B_{n\times n}$ such that

$$AB = BA = I$$

where $I_{n\times n}$ is the identity matrix. In this case the matrix B is called the inverse matrix of A denoted by A^{-1}. The Identity matrix it can be defined by adjoint matrix.

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

adjoint of a matrix A is defined by matrix of cofactors C_{ij} as follows

$$\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

The matrix inverse is written terms of determinant and thus if $det(A) = 0$ then the matrix A is said to be non invertible or singular.

Example :

Conseder the matrix $A = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix}$. The determinant is $det(A) = 45 \neq 0$ then the matrix A is invertible. Let us calculate the cofactors

$$C_{11} = (-1)^{(1+1)} M_{11} = \begin{vmatrix} 0 & 7 \\ -1 & 4 \end{vmatrix} = 7$$
$$C_{12} = -\begin{vmatrix} 3 & 7 \\ 4 & 4 \end{vmatrix} = 16,\ C_{13} = \begin{vmatrix} 3 & 0 \\ 4 & -1 \end{vmatrix} = -3$$
$$C_{21} = -\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = -6\ C_{22} = \begin{vmatrix} 5 & 2 \\ 4 & 4 \end{vmatrix} = 12$$
$$C_{23} = -\begin{vmatrix} 5 & 1 \\ 4 & -1 \end{vmatrix} = 9\ C_{31} = \begin{vmatrix} 1 & 2 \\ 0 & 7 \end{vmatrix} = 7$$
$$C_{32} = -\begin{vmatrix} 5 & 2 \\ 3 & 7 \end{vmatrix} = -29\ C_{33} = \begin{vmatrix} 5 & 1 \\ 3 & 0 \end{vmatrix} = 3$$

Then we get $adj(A) = \begin{pmatrix} 7 & 16 & -3 \\ -6 & 12 & 9 \\ 7 & -29 & 3 \end{pmatrix}^T = \begin{pmatrix} 7 & -6 & 7 \\ 16 & 12 & -29 \\ -3 & 9 & 3 \end{pmatrix}$ and this gives

$$A^{-1} = \frac{1}{45} \begin{pmatrix} 7 & -6 & 7 \\ 16 & 12 & -29 \\ -3 & 9 & 3 \end{pmatrix}$$

Properties :

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = A^{-1}B^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

7 Rank of a matrix :

The rank of a matrix is the dimension of the vector space generated by its columns. Thus, the rank is just the maximal number of linearly independent columns or rows.

Key properties :

- If A is a $n \times n$ matrix and $det(A) \neq 0$, then $rank(A) = n$.
- If $det(A) = 0$, then $rank(A) < n$.
- if A is a nonzero matrix then $1 \leq rank(A) \leq n$
- $rank(A) = rank(A^T)$

Example :

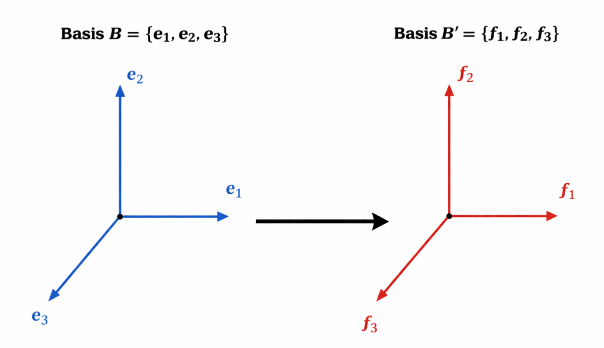
The rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 6 \end{pmatrix}$ is less than 3, since the first and second rows are linearly dependent $R_2 = 2R_1$, also we have $det(A) = 0$.

8 Change of basis

In linear algebra, a basis is defined as a set of linearly independent vectors that span a given vector space. Moreover, a vector space generally admits infinitely many possible bases. The representation of any vector is always given relative to a chosen basis, and its coordinates depend on this choice. Although the coordinate representation may vary from one basis to another, the vector itself remains geometrically invariant. The concept of change of basis is therefore a fundamental tool, as it allows one to simplify problems and reduce computational complexity by selecting a more convenient representation.

8.1 Definition

Let V be a three dimensional vector space over a field of scalars F and let $\mathcal{B} = \{e_1, e_2, e_3\}$and $\mathcal{B}' = \{f_1, f_2, f_3\}$be two bases for V.



A vector v with its coordinates (x_1, x_2, x_3) in terms of the basis $\mathcal{B}$ it can be shown as follows :

$$[v]_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1e_1 + x_2e_2 + x_3e_3 = \sum_{i=1}^{3} x_ie_i$$

The same vector v with its coordinates (x', y', z') in terms of the basis $\mathcal{B}'$ it can be written as :

$$[v]_{B'} = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = x'_1f_1 + x'_2f_2 + x'_3f_3 = \sum_{i=1}^{3} x'_if_i$$

A change of basis can be interpreted as a transformation that expresses the vector v in terms of the coordinates associated with another basis. the vector itself remains invariant. This invariance is expressed by the equality of its representations in different bases :

$$\boxed{\sum_{i=1}^{3} x_i e_i = \sum_{i=1}^{3} x'_i f_i}$$

More generally, in an n-dimensional vector space, one has :

$$\boxed{\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x'_i f_i}$$

where $\{e_1, e_2, ..., e_n\}$and $\{f_1, f_2, ..., f_n\}$ are two different bases of the same vector space, and (x_i), (x'_i) are the corresponding coordinate representations of the same vector v.

Example.

Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$and $\mathcal{B}' = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$be two bases of $\mathbb{R}^3$, and let $[v]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$be a vector in $\mathbb{R}^3$written tems of the basis $\mathcal{B}$. In order to define this vector in terms of the new basis $\mathcal{B}'$ we use the expression

$$\sum_{i=1}^{3} x_i e_i = \sum_{i=1}^{3} x'_i f_i$$

then we can write

$$3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = x'_1 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + x'_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x'_3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

and thus we get

$$\begin{aligned} 2x'_1 + x'_2 &= 4 \\ x'_2 + x'_3 &= 7 \\ 3x'_3 &= 5 \end{aligned} .$$

Solving the system gives

$$x'_3 = \frac{5}{3}, \quad x'_2 = \frac{16}{3}, \quad x'_1 = -\frac{2}{3}$$

Finally, we obtain the expression for the vector v relative to the basis $\mathcal{B}'$ which is written as follows

$$[v]_{\mathcal{B}'} = \frac{1}{3}\begin{pmatrix} -2 \\ 16 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{-2}{3} \\ \frac{16}{3} \\ \frac{5}{3} \end{pmatrix}$$

8.2 The matrix of change of basis

To determine the matrix S that transforms the basis $\mathcal{B} = \{e_1, e_2, e_3\}$into $\mathcal{B}' = \{f_1, f_2, f_3\}$, we express each basis vector e_i as a linear combination of the vectors f_i so

$$e_j = \sum_{i=1}^{3} s_{ij} f_i$$

where s_{ij} are the coeffcients of the matrix S. Also the matrix S is called the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$ and denoted by $S_{\mathcal{B}\to\mathcal{B}'}$. Through the use of this matrix, we can find the coordinates of this vector in the new basis $\mathcal{B}'$as follows :

$$[v]_{\mathcal{B}'} = S_{\mathcal{B}\to\mathcal{B}'}[v]_{\mathcal{B}}$$

Simmilarly, the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$ is defined by

$$f_j = \sum_{i=1}^{3} s_{ij} e_i$$

where s_{ij} are the coeffcients of the matrix $S_{\mathcal{B}'\to\mathcal{B}}$, and thus we write

$$[v]_{\mathcal{B}} = S_{\mathcal{B}'\to\mathcal{B}}[v]_{\mathcal{B}'}$$

Example.

Let us consider the same situation as in the previous example. The change-of-basis matrix from $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$to $\mathcal{B}' = \left\{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right\}$can be determined as follows :

- The expression $e_j = \sum_{i=1}^{3} s_{ij} f_i$ gives

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = s_{11}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s_{21}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s_{31}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = s_{12}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s_{22}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s_{32}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = s_{13}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s_{23}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s_{33}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

This leads to a three systems of linear equations, which can be written as follows :

— The first system :

$$\begin{aligned} 2s_{11} + s_{21} &= 1 \\ s_{21} + s_{31} &= 1 \\ 3s_{31} &= 0 \end{aligned}$$

— The second system :

$$\begin{aligned} 2s_{12} + s_{22} &= 1 \\ s_{22} + s_{32} &= 0 \\ 3s_{32} &= 1 \end{aligned}$$

— The third system

$$\begin{aligned} 2s_{13} + s_{23} &= 0 \\ s_{23} + s_{33} &= 1 \\ 3s_{33} &= 1 \end{aligned}$$

- The solutions are given by :

$$s_{11} = s_{31} = 0,\ s_{21} = 1,\ s_{32} = s_{33} = -s_{22} = -s_{13} = \frac{1}{3},\ s_{12} = s_{23} = \frac{2}{3}$$

Consequently, the change-of-basis matrix is obtained as follows :

$$S_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Applaying this matrix on a vector $[v]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, we get the same result as we see before

$$[v]_{\mathcal{B}'} = S_{\mathcal{B}\to\mathcal{B}'}[v]_{\mathcal{B}} = \frac{1}{3}\begin{pmatrix} 0 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ 16 \\ 5 \end{pmatrix}$$

The student can take the opposite case of this example, where he looks for the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$, then takes the vector $[v]_{\mathcal{B}'} = \frac{1}{3}\begin{pmatrix} -2 \\ 16 \\ 5 \end{pmatrix}$ and applies the founded matrix $S_{\mathcal{B}\to\mathcal{B}'}$ to it to verify the previous relationships himself.
The reader may also consider the reverse situation of this example, namely determining the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$. Then, by taking the vector

$$[v]_{\mathcal{B}'} = \frac{1}{3}\begin{pmatrix} -2 \\ 16 \\ 5 \end{pmatrix}$$

and applying the obtained matrix $S_{\mathcal{B}'\to\mathcal{B}}$, one can verify the consistency of the preceding results.