

Subject : *Algebra 4*
Instructor : *Y. Halim*

SERIES OF EXERCISES NO. 3

Exercise 1 : (Exam 2024)

Let E be a vector space over a field k , q a quadratic form on E and f its associated polar form.

1. Prove the identity

$$\forall (x, y) \in E \times E : f(x, y) = \frac{1}{4} [q(x + y) - q(x - y)].$$

2. Deduce the following equivalence

$$q(x) = q(y) \iff f(x + y, x - y) = 0.$$

Exercise 2 :

Let q be a quadratic form defined on \mathbb{R}^3 by

$$q((x_1, x_2, x_3)) = x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + x_2x_3.$$

1. Determine the polar form associated with q .
2. Determine the matrix of q in the canonical basis of \mathbb{R}^3 .
3. Determine $\ker(q)$ (the kernel of q) and $rg(q)$ (the rank of q).
4. Is q non-degenerate?

Exercise 3 :

Let the matrix in $\mathcal{M}_3(\mathbb{R})$ be

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 5 & 3 & -3 \\ 0 & -3 & 8 \end{pmatrix}.$$

1. Determine q , the associated quadratic form.
2. Determine f , the symmetric bilinear form defined on \mathbb{R}^3 whose associated matrix in the canonical basis is A .
3. Let

$$B' = \{(1, 0, 0), (-1, 1, 0), (-1, 1, 1)\}$$

be a basis of \mathbb{R}^3 .

— Show that B' is an orthogonal basis.

4. Determine F^\perp where

$$F = \{(x, y, z) \in \mathbb{R}^3 : x + y = 0, z = 0\}.$$

Exercise 4 : (Test 2022)

Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

1. Determine f , the symmetric bilinear form defined on \mathbb{R}^3 whose associated matrix in the canonical basis is A .
2. Determine q , the associated quadratic form.
3. Show that q is non-degenerate.

Exercise 5 :

The following quadratic forms are defined on \mathbb{R}^3 :

1. $q_1(X) = x_1^2 - x_2^2 + 2x_2x_3 - 4x_1x_3$.
2. $q_2(X) = 3x_1x_2 + 2x_1x_3 + x_2x_3$.
3. $q_3(X) = x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + x_2x_3$.

Determine the Gauss reductions and deduce the signature, the rank and an orthogonal basis for each one.

Exercise 6 :

Let q be a quadratic form defined on \mathbb{R}^3 by

$$q((x_1, x_2, x_3)) = x_1x_2 - x_2x_3 + 2x_1x_3.$$

1. Determine the matrix of q in the canonical basis of \mathbb{R}^3 and deduce its rank.
2. Show that

$$\varepsilon_1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \quad \varepsilon_2 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right), \quad \varepsilon_3 = (1, 2, 1)$$

form an orthogonal basis of \mathbb{R}^3 .

3. Deduce a reduced form of q and determine its signature $sg(q)$.
4. Determine an isotropic vector for q .

Exercise 7 : (Exam 2024)

Let f_α be the bilinear form defined on $\mathbb{R}^3 \times \mathbb{R}^3$ by

$$f_\alpha((x_1, x_2, x_3), (y_1, y_2, y_3)) = \alpha x_1 y_1 + 2x_2 y_2 - \alpha x_3 y_3 - x_1 y_3 - y_1 x_3.$$

1. Determine q_α , the quadratic form associated with f_α .
2. Using Gauss' algorithm, decompose q_α into a sum of squares.
3. According to the values of the parameter α , determine the rank and the signature of q_α .
4. Assume $\alpha \neq 0$. Determine an orthogonal basis for q_α .

Exercise 8 : (Exam 2016)

Let q_α be the quadratic form defined on \mathbb{R}^3 by

$$q_\alpha(x_1, x_2, x_3) = x_1^2 + (1 + \alpha)x_2^2 + (1 + \alpha + \alpha^2)x_3^2 + 2x_1x_2 - 2\alpha x_2x_3.$$

1. Determine the matrix A_α associated with q_α in the canonical basis of \mathbb{R}^3 .
2. Determine the values of α for which q_α is non-degenerate.
3. Give a Gauss reduction of q_α .
4. According to the values of α , determine the signature of q_α .
5. Determine an orthogonal basis for q_α .