Abdelhafid Boussouf University of Mila
Faculty of Mathematics and Computer Science
Second Year LMD Mathematics and Applied Mathematics 2025/2026

Subject : *Algebra 4*
Instructor : *Y. Halim*

Series of Exercises No. 2

Exercise 1 :
Show that the following mappings are bilinear forms. Are they symmetric or skew-symmetric ?

1.

$$\begin{array}{rcl} \Psi : & \mathbb{R}_n[X] \times \mathbb{R}_n[X] & \to \mathbb{R} \\ & (P, Q) & \mapsto \displaystyle\int_0^2 x^2 P(x) Q'(x) dx \end{array}$$

2.

$$\begin{array}{rcl} f : & M_n(\mathbb{R}) \times M_n(\mathbb{R}) & \to \mathbb{R} \\ & (A, B) & \mapsto Tr(AB) \end{array}$$

Exercise 2 :
Let Ψ be the mapping defined on $\mathbb{R}_2[X]$ by

$$\begin{array}{rcl} \Psi : & \mathbb{R}_2[X] \times \mathbb{R}_2[X] & \to \mathbb{R} \\ & (P, Q) & \mapsto \displaystyle\int_0^1 P(x) Q(1-x) dx \end{array}$$

1. Show that Ψ is a symmetric bilinear form.
2. Determine the matrix of Ψ in the canonical basis of $\mathbb{R}_2[X]$.
3. Show that Ψ is non-degenerate.

Exercise 3 :
Let the matrix

$$M = \begin{pmatrix} 0 & -3 & 2 \\ -3 & 5 & 7 \\ 2 & 7 & -1 \end{pmatrix}$$

1. Justify that the matrix A is symmetric.
2. Determine the symmetric bilinear form defined on $\mathbb{R}^3$ whose associated matrix in the canonical basis is the matrix A.
3. Determine the associated quadratic form.

Exercise 4 :
Consider the following bilinear forms.

$$\begin{array}{llll} f: & \mathbb{R}^2 \times \mathbb{R}^2 & \rightarrow \mathbb{R} \\ & ((x_1, x_2), (y_1, y_2)) & \mapsto x_1 y_1 + 2x_2 y_2 - 4x_2 y_1 - 4x_1 y_2 \end{array}$$

$$\begin{array}{llll} g: & \mathbb{R}^3 \times \mathbb{R}^3 & \rightarrow \mathbb{R} \\ & ((x_1, x_2, x_3), (y_1, y_2, y_3)) & \mapsto 3x_1 y_1 - 5x_2 y_2 - 2x_1 y_2 - 2x_2 y_1 + 3x_2 y_3 + 3y_2 x_3 \end{array}$$

1. Show that f and g are symmetric.
2. Determine the quadratic form associated with each bilinear form.

Exercise 5 : (Exam 2023)
Let the mapping

$$\begin{array}{llll} f: & \mathbb{R}^2 \times \mathbb{R}^2 & \rightarrow \mathbb{R} \\ & ((x_1, x_2), (y_1, y_2)) & \mapsto 2x_1 y_2 + 2x_2 y_1 - 5x_2 y_2 \end{array}$$

1. Show that f is a symmetric bilinear form.
2. Determine $Ann(f)$, the kernel (annihilator) of f. Deduce whether f is non-degenerate.
3. Determine the matrix of f in the basis $\{\varepsilon_1 = (1, 1), \varepsilon_2 = (1, -3)\}$ of $\mathbb{R}^2$.
4. Let the set
$$F =< (2, 1) >$$
Determine $F^{\perp}$, the orthogonal complement of F.

Bibliographic References :

1. Exercices corrigés d'algèbre linéaire, **Tom 2**, 510/27.
2. Dualité, formes quadratiques, formes hermitiennes : exercices corrigés avec rappels de cours, 510/12.
3. Algèbre linéaire et bilinéaire, 510/516.
4. Algèbre et géométrie, 2 année, 510/1058.
5. Algèbre linéaire : Cours et exercices corrigés 510/24.
6. Algèbre, exercices et problèmes, 510/420.