Algebra IV

Course 2025-2026

Yacine Halim

Contents

1 Linear Forms, Duality 5

1.1 Linear Forms 5

1.2 Hyperplanes 6

1.3 Duality 7

1.3.1 Dual Vector Space 7

1.3.2 Dual Basis 8

1.3.3 Bidual of a Vector Space 9

2 Bilinear Forms on a Vector Space 11

2.1 Bilinear Forms 11

2.1.1 The Vector Space $\mathscr{L}(E \times F, \mathbb{K})$ 13

2.1.2 Matrix of a Bilinear Form 13

2.1.3 Bilinear Form Associated with a Matrix 15

2.1.4 Change of Basis 15

2.2 Symmetric Bilinear Forms 16

2.2.1 Kernel of a Symmetric Bilinear Form 19

2.2.2 Nondegenerate Symmetric Bilinear Forms 19

2.3 Orthogonality 21

2.3.1 Orthogonal Bases 21

2.4 Orthogonal matrices 21

2.4.1 Properties of Orthogonal Matrices 21

2.5 Examples 22

3 Quadratic Forms 23

3.1 Quadratic Forms 23

3.1.1 Properties of Quadratic Forms 24
3.1.2 Isotropic Cone 25
3.1.3 Kernel of a Quadratic Form 25
3.1.4 Matrix Associated with a Quadratic Form 25
3.1.5 Quadratic Form Associated with a Square Matrix 26
3.1.6 Rank of a Quadratic Form 28

3.2 Norm Associated with an Inner Product 33

4 Introduction to Hermitian Spaces 35

4.1 Introduction 35

4.2 Preliminary Notions 35

4.2.1 Semilinear Maps and Sesquilinear Forms 35
4.2.2 Conjugate Matrix, Adjoint Matrix 37
4.2.3 Normal Matrices 39
4.2.4 Hermitian Matrices 40

4.3 Hermitian Space 42

4.3.1 Hermitian Inner Product 42
4.3.2 Expressions in a Basis 43

Bibliography 45

Books 45

Articles 45

1. Linear Forms, Duality

Introduction

This chapter introduces the notion of linear forms and the concept of duality in vector spaces. These notions play an important role in linear algebra and appear naturally in many mathematical problems.

A linear form is a linear mapping from a vector space into the base field $\mathbb{R}$ or $\mathbb{C}$. Linear forms allow us to describe hyperplanes and provide a useful tool for studying vector spaces from a new point of view.

We then define the dual space of a vector space as the set of all linear forms on that space. In the finite-dimensional case, the dual space has the same dimension as the original space, and each basis admits a corresponding dual basis.

Finally, we introduce the bidual space and show that every finite-dimensional vector space can be identified in a natural way with its bidual. This chapter provides the basic tools needed for the study of more advanced topics in linear algebra and related fields.

1.1 Linear Forms

In the following, E denotes a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, of finite or infinite dimension.

Definition 1.1.1 A linear form on E is a linear mapping from E into K.

■ **Example 1.1** ■

.

1.2 Hyperplanes

Hyperplanes are vector subspaces defined as kernels of nonzero linear forms. In finite-dimensional spaces, a hyperplane is a subspace whose dimension is one less than that of the ambient space.

Proposition 1.2.1 Let f be a nonzero linear form on a vector space E of dimension n. Then

$$\dim \ker f = n - 1.$$

Proof. ■

Proposition (1.2.1) leads to a generalization of the notion of hyperplane to vector spaces of finite or infinite dimension.

Definition 1.2.1 A hyperplane of E is a vector subspace $H \subset E$ such that there exists a nonzero linear form $f : E \to K$ satisfying $H = \ker f$.

If $\dim E = n < \infty$, Proposition (1.2.1) recovers the classical definition of a hyperplane: a subspace of dimension $n-1$.

■ **Example 1.2** ■

Corollary 1.2.2 Let H be a hyperplane of E. Then for any one-dimensional vector subspace D of E not contained in H, we have

$$E = H \oplus D.$$

R By definition, a one-dimensional vector subspace is called a vector line.

1.3 Duality

Duality studies a vector space through the linear forms defined on it. The dual space consists of all linear forms, and in finite dimension it has the same dimension as the original space, with a natural associated dual basis.

1.3.1 Dual Vector Space

Definition 1.3.1 The dual vector space of E, denoted E^*, is the vector space of all linear forms on E:

$$E^* = \mathscr{L}(E, K).$$

Corollary 1.3.1 If E is finite-dimensional, then E^* is also finite-dimensional and

$$\dim E = \dim E^*.$$

Proof. ■

1.3.2 Dual Basis

Assume E is finite-dimensional and let $\mathbf{B} = \{e_1, e_2, \ldots, e_n\}$ be a basis of E. Define the family $\mathbf{B}^* = \{e_1^*, e_2^*, \ldots, e_n^*\}$ in E^* by

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.1}$$

Theorem 1.3.2 Let E be a vector space of dimension n. Then its dual E^* has dimension n, and for any basis $\mathbf{B} = \{e_1, e_2, \ldots, e_n\}$ of E, the family $\mathbf{B}^* = \{e_1^*, e_2^*, \ldots, e_n^*\}$ is a basis of E^*, called the dual basis of $\mathbf{B}$.

Proof. ■

■ **Example 1.3** ■

1.3.3 Bidual of a Vector Space

Definition 1.3.2 Let E be a vector space over K. The bidual of E, denoted E^{**}, is the dual of E^{*}:

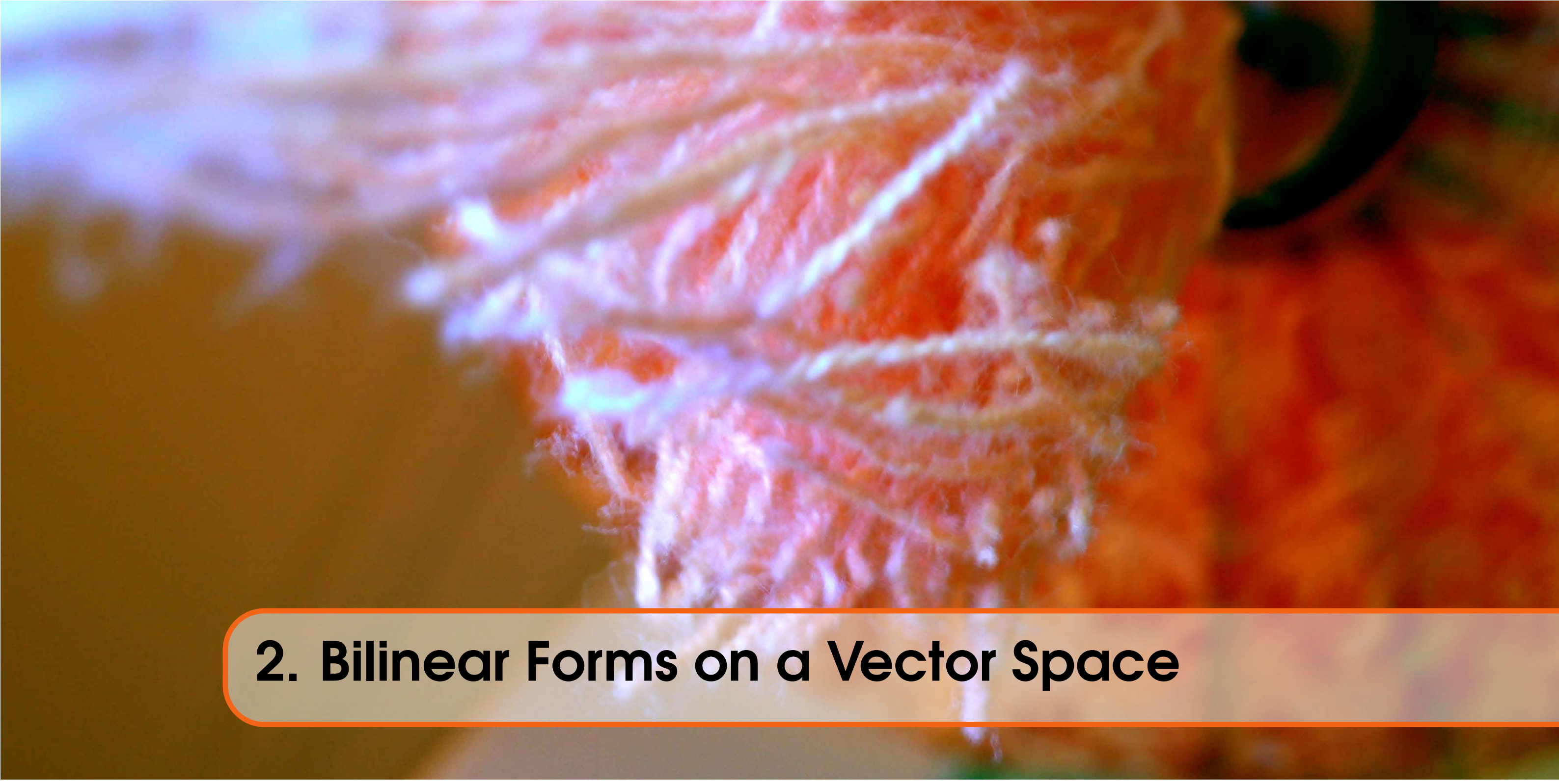
$$E^{**} = (E^{*})^{*} = \mathscr{L}(E^{*}, K).$$

Proposition 1.3.3 If E is finite-dimensional, then the mapping

$$\begin{aligned} \widetilde{\ }: \quad E &\longrightarrow E^{**} \\ x &\longmapsto \widetilde{x} \end{aligned}$$

defined by $\widetilde{x}(f) = f(x)$ for all $f \in E^{*}$ is a linear bijection. This mapping is called the **canonical isomorphism** between E and E^{**}.

Proof. ■



2. Bilinear Forms on a Vector Space

Throughout this chapter, E, F, and G are vector spaces over the same field $\mathbb{K}$.

2.1 Bilinear Forms

Definition 2.1.1 A mapping $f : E \times F \longrightarrow G$ is said to be **bilinear** if:

$$\begin{aligned} f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y) \\ f(\alpha x, y) &= \alpha f(x, y) \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2) \\ f(x, \alpha y) &= \alpha f(x, y) \end{aligned}$$

for all $x, x_1, x_2 \in E$, $y, y_1, y_2 \in F$ and $\alpha \in \mathbb{K}$.

If $G = \mathbb{K}$, f is called a **bilinear form**.

Definition 2.1.2 A mapping $f : E \times F \longrightarrow G$ is said to be bilinear if:

i) For every $x \in E$ (with x fixed), the mapping

$$\begin{aligned} f_x : \quad F &\longrightarrow G \\ y &\longmapsto f(x, y) \end{aligned}$$

is a linear form on F.

ii) For every $y \in F$ (with y fixed), the mapping

$$\begin{aligned} f_y : \quad E &\longrightarrow G \\ x &\longmapsto f(x, y) \end{aligned}$$

is a linear form on E.

■ **Example 2.1** ■

2.1.1 The Vector Space $\mathscr{L}(E \times F, \mathbb{K})$

Proposition 2.1.1 The set of all bilinear forms from $E \times F$ into $\mathbb{K}$, denoted $\mathscr{L}(E \times F, \mathbb{K})$, is a vector space over $\mathbb{K}$.

Proof. ■

.

2.1.2 Matrix of a Bilinear Form

Let E and F be finite-dimensional vector spaces over the same field $\mathbb{K}$, $B = \{e_1, e_2, \ldots, e_n\}$ a basis of E and $B' = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p\}$ a basis of F.

Definition 2.1.3 Let $f \in \mathscr{L}(E \times F, \mathbb{K})$. The **matrix associated with** f with respect to B and B' is the (n, p) matrix defined by

$$a_{ij} = f(e_i, \varepsilon_j), \quad i = 1, \ldots, n, \; j = 1, \ldots, p,$$

in other words

$$M = \begin{pmatrix} f(e_1,\varepsilon_1) & f(e_1,\varepsilon_2) & \ldots & f(e_1,\varepsilon_p) \\ f(e_2,\varepsilon_1) & f(e_2,\varepsilon_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ f(e_n,\varepsilon_1) & f(e_n,\varepsilon_2) & \ldots & f(e_n,\varepsilon_p) \end{pmatrix}.$$

■ **Example 2.2** ■

.

2.1.3 Bilinear Form Associated with a Matrix

Definition 2.1.4 Let $M \in \mathscr{M}_{n,p}(\mathbb{K})$. The mapping

$$\begin{array}{rcl} f : \mathbb{K}^n \times \mathbb{K}^p & \longrightarrow & \mathbb{K} \\ (X,Y) & \longmapsto & X^t M Y \end{array}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ et $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$ is a bilinear form called the **bilinear form associated with** M.

■ **Example 2.3** ■

2.1.4 Change of Basis

Proposition 2.1.2 Let $f \in \mathscr{L}(E \times F, \mathbb{K})$. If M and N are the matrices of f with respect to bases (B, B') and (B_1, B'_1), then

$$N = {}^t PMQ,$$

where P and Q are the change-of-basis matrices.

Proof. ■

.

2.2 Symmetric Bilinear Forms

Definition 2.2.1 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$.

- f is called **symmetric** if

$$\forall (x,y) \in E \times E, \quad f(x,y) = f(y,x).$$

- f is called **skew-symmetric** if

$$\forall (x,y) \in E \times E, \quad f(x,y) = -f(y,x).$$

- f is called **alternating** if

$$\forall x \in E, \quad f(x,x) = 0.$$

■ **Example 2.4**

■

.

Proposition 2.2.1

i) The set of symmetric bilinear forms, denoted $\mathscr{S}(E\times E,\mathbb{K})$, is a vector subspace of $\mathscr{L}(E\times E,\mathbb{K})$.
ii) The set of skew-symmetric bilinear forms, denoted $\mathscr{A}(E\times E,\mathbb{K})$, is a vector subspace of $\mathscr{L}(E\times E,\mathbb{K})$.

Proof.

■

Proposition 2.2.2 We have

$$\mathscr{L}(E \times E, \mathbb{K}) = \mathscr{S}(E \times E, \mathbb{K}) \oplus \mathscr{A}(E \times E, \mathbb{K}).$$

Proof.

■

2.2.1 Kernel of a Symmetric Bilinear Form

Definition 2.2.2 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$. The **kernel** of f, denoted $Ann(f)$, is

$$Ann(f) = \{x \in E : f(x,y) = 0, \forall y \in E\}.$$

2.2.2 Nondegenerate Symmetric Bilinear Forms

Definition 2.2.3 f is said to be **nondegenerate** if $Ann(f) = \{0_E\}$.

■ **Example 2.5** ■

Proposition 2.2.3 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$. Consider the map

$$\Phi : E \longrightarrow E^* \quad \text{defined by} \quad \forall y \in E,\ \Phi(y) = \varphi_y,\ \text{where } \forall x \in E,\ \varphi_y(x) = f(x,y).$$

Then Φ is injective if and only if f is non-degenerate.

Proof. ■

Corollary 2.2.4 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$, where E is finite-dimensional, B is a basis of E, and A is the matrix of f with respect to the basis B. Then

$$f \text{ is non-degenerate } \Leftrightarrow \det(A) \neq 0.$$

Proof. ■

2.3 Orthogonality

2.3.1 Orthogonal Bases

Definition 2.3.1 A family $\mathscr{F}$ of E is said to be *orthogonal* for f (or simply orthogonal if there is no ambiguity about f) if for all $x, y \in \mathscr{F}$, with $x \neq y$, we have $x \perp_f y$. In other words, if

$$f(x,y) = 0_K \qquad (\forall x, y \in \mathscr{F},\ x \neq y).$$

— When a basis $\mathscr{B}$ of E is an orthogonal family for f, we say that $\mathscr{B}$ is an *orthogonal basis* of E for f (or simply an orthogonal basis of E if there is no ambiguity about f).

— The expression "orthogonal for f" is sometimes replaced by one of the following expressions: "orthogonal for q_f", "f-orthogonal", or "q_f-orthogonal".

When E is finite-dimensional, determining an orthogonal basis of E for f is very useful for simplifying and classifying the symmetric bilinear form f; moreover, such a basis always exists!

Theorem 2.3.1 Suppose that E is finite-dimensional. Then there exists at least one orthogonal basis of E for f.

2.4 Orthogonal matrices

Definition 2.4.1 A square matrix $A \in \mathbb{R}^{n\times n}$ is called **orthogonal** if

$$A^T A = AA^T = I_n,$$

where A^T denotes the transpose of A and I_n is the $n \times n$ identity matrix.

Equivalently, a matrix is orthogonal if its columns (or rows) form an orthonormal set of vectors with respect to the standard Euclidean inner product.

2.4.1 Properties of Orthogonal Matrices

Orthogonal matrices possess several important properties:

Proposition 2.4.1 Let $A \in \mathbb{R}^{n\times n}$ be orthogonal. Then:

1. $A^{-1} = A^T$ (the inverse equals the transpose).
2. $\det(A) = \pm 1$.
3. Orthogonal matrices preserve the Euclidean norm: for any $x \in \mathbb{R}^n$,

$$||Ax|| = ||x||.$$

4. The columns (and rows) of A form an orthonormal basis of $\mathbb{R}^n$:

$$\langle a_i, a_j \rangle = \delta_{ij}.$$

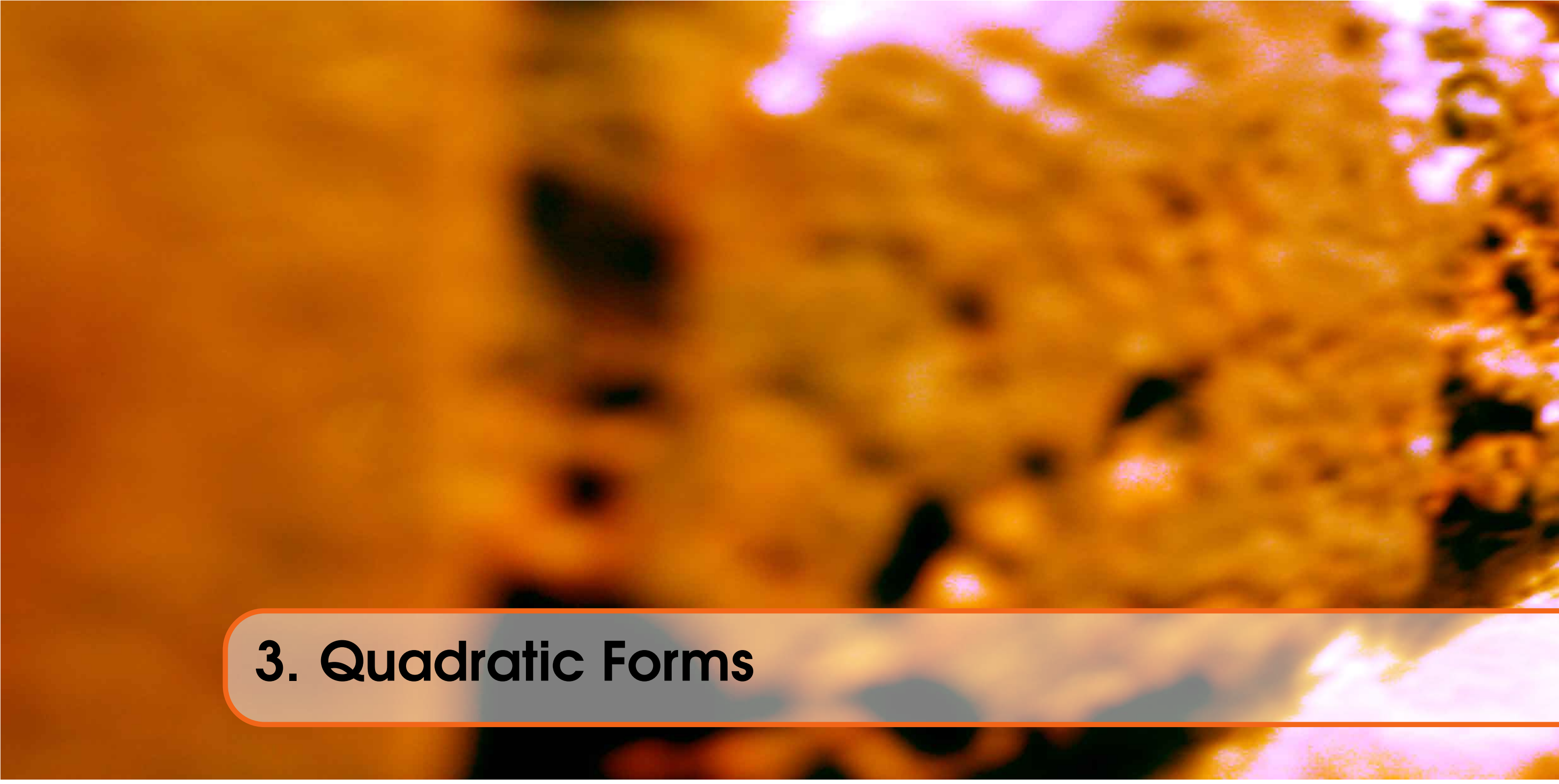
Theorem 2.4.2 — Preservation of Inner Product. If A is orthogonal and $x, y \in \mathbb{R}^n$, then

$$\langle Ax, Ay \rangle = \langle x, y \rangle.$$

Proof. ■

2.5 Examples

■ **Example 2.6** ■



3. Quadratic Forms

3.1 Quadratic Forms

Definition 3.1.1 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$. The mapping

$$\begin{aligned} q: \quad E &\longrightarrow \mathbb{K} \\ x &\longmapsto q(x) = f(x,x) \end{aligned}$$

is called the **quadratic form associated with** f.

The mapping f is called the **polar form associated with** q.

■ **Example 3.1** ■

3.1.1 Properties of Quadratic Forms

Proposition 3.1.1 Let q be a quadratic form on E, a vector space over $\mathbb{K}$, and let f be its polar form.

i) $q(\alpha x) = \alpha^2 q(x), \quad \forall x \in E, \forall \alpha \in \mathbb{K}$.

ii) $q(x+y) = q(x) + q(y) + 2f(x,y), \quad \forall x,y \in E$ (Polarization identity).

iii) $q(x+y) + q(x-y) = 2(q(x)+q(y)), \quad \forall x,y \in E$ (Parallelogram identity).

Proof. ■

R The polarization identity allows one to recover f from q:

$$f(x,y) = \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right].$$

■ **Example 3.2** ■

3.1.2 Isotropic Cone

Definition 3.1.2 Let q be a quadratic form on E, a vector space over $\mathbb{K}$. A vector $x \in E$ is called an **isotropic vector** if $q(x) = 0$.

The set of all isotropic vectors of q, denoted by $C(q)$, is called the **isotropic cone**, that is,

$$C(q) = \{x \in E \; ; \; q(x) = 0\}.$$

R

1. Since $q(0) = 0$, the zero vector is always isotropic.
2. In general, the isotropic cone is not a vector subspace of E.

■ **Example 3.3** ■

3.1.3 Kernel of a Quadratic Form

Definition 3.1.3 The **kernel** of a quadratic form, denoted by $N(q)$, is the kernel of its polar form. That is,

$$N(q) = \mathrm{Ann}(f) = \{x \in E \; ; \; f(x,y) = 0 \; \forall y \in E\}.$$

3.1.4 Matrix Associated with a Quadratic Form

Definition 3.1.4 Let q be a quadratic form on E, a finite-dimensional vector space over $\mathbb{K}$, and let B be a basis of E. The matrix associated with q with respect to the basis B, denoted by $\mathscr{M}_B(q)$, is the matrix associated with the polar form f of q with respect to B, that is,

$$\mathscr{M}_B(q) = \mathscr{M}_B(f).$$

R The matrix associated with a quadratic form is symmetric, that is,

$$\mathscr{M}_B(q) = {}^t\mathscr{M}_B(q).$$

Definition 3.1.5 Let q be a quadratic form on $\mathbb{R}^n$ and let $B = \{e_1, e_2, \dots, e_n\}$ be the canonical basis. The matrix associated with q with respect to B is

$$M_B(q) = \frac{1}{2}\begin{pmatrix} \frac{\partial q}{\partial x_1}(e_1) & \frac{\partial q}{\partial x_2}(e_1) & \cdots & \frac{\partial q}{\partial x_n}(e_1) \\ \frac{\partial q}{\partial x_1}(e_2) & \frac{\partial q}{\partial x_2}(e_2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q}{\partial x_1}(e_n) & \frac{\partial q}{\partial x_2}(e_n) & \cdots & \frac{\partial q}{\partial x_n}(e_n) \end{pmatrix}.$$

Example 3.4 ■

3.1.5 Quadratic Form Associated with a Square Matrix

Proposition 3.1.2 Let $M = (a_{ij})$ be a symmetric matrix associated with the quadratic form q with respect to the canonical basis $\{e_1, e_2, \dots, e_n\}$ of E. Then

$$\forall x = \sum_{i=1}^{n} x_i e_i, \quad q(x) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{1 \le i < j \le n} a_{ij} x_i x_j.$$

Proof. ■

.

.

R The diagonal entries of M are the coefficients of the terms x_i^2.

■ **Example 3.5** ■

3.1.6 Rank of a Quadratic Form

Definition 3.1.6 The **rank** of a quadratic form q, denoted by $\operatorname{rank}(q)$, is the rank of its associated matrix with respect to any basis of E.

Reduction of Quadratic Forms via Gauss Method

Gauss Reduction Process

In such a basis, the matrix associated with q relative to $\mathscr{B}$ is diagonal. If this matrix is denoted by

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (\text{with } \lambda_1, \lambda_2, \ldots, \lambda_n \in K),$$

then for every $x = x_1u_1 + x_2u_2 + \cdots + x_nu_n \in E$, we have

$$q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

Such a representation of q (i.e., a linear combination of squares of K-linearly independent linear forms) is called a *reduction* (or a reduced form) of q.

Starting from the expression of q relative to an arbitrary basis of E, we shall establish in what follows an efficient algorithm (due to Gauss) that allows one to obtain (in a finite number of steps) a reduced form of q.

Let $\mathscr{B} = (e_1, e_2, \ldots, e_n)$ be a basis of E. According to Proposition 3.2, the expression of q relative to $\mathscr{B}$ can be written (for every $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$, with $x_1, x_2, \ldots, x_n \in K$) as

$$q(x) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j,$$

where the a_{ij} are scalars (i.e., elements of K).

The Gauss algorithm for reducing q is a recursive procedure. Its main purpose is to show how to write $q(x)$ as a linear combination of the square of a linear form and a homogeneous polynomial of degree two in $(n-1)$ other linear forms.

The rest is clear: it suffices to repeat this process as many times as necessary until $q(x)$ is expressed as a linear combination of squares of linear forms.

The algorithm guarantees the linear independence of the linear forms that appear at each step; in particular, it ensures the linear independence of the linear forms appearing in the final reduction. These linear forms may therefore be regarded as coordinates relative to a new basis of E. This will precisely be the orthogonal basis corresponding to the reduced form of q.

4.1.1 Description of Gauss's Algorithm

- If $n = 1$, then the quadratic form q is already reduced and there is nothing to do.
- Assume from now on that $n > 2$. We distinguish the following two cases:
  First case: (if the a_{ii} are not all zero).
  Up to permuting[1] the coordinates $x_1, x_2, \ldots, x_n$, one may assume that $a_{11} \neq 0$. We then write $q(x)$ (for $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$) as a polynomial of degree two in x_1 (with coefficients in $K[x_2, x_3, \ldots, x_n]$), and then put it into canonical form. More precisely, for every $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$, we have:

[1] A permutation of the coordinates corresponds to a permutation of the vectors of the basis of E under consideration.

$$
\begin{aligned}
q(x) &= a_{11}x_1^2 + \sum_{2\le j\le n} a_{1j}x_1x_j + \sum_{2\le i,j\le n} a_{ij}x_ix_j \\
&= a_{11}x_1^2 + \left(\sum_{2\le j\le n} a_{1j}x_j\right)x_1 + \sum_{2\le i,j\le n} a_{ij}x_ix_j \\
&= a_{11}\left(x_1^2 + \frac{1}{a_{11}}\sum_{2\le j\le n} a_{1j}x_jx_1\right) + \sum_{2\le i,j\le n} a_{ij}x_ix_j \\
&= a_{11}\left[\left(x_1 + \frac{1}{2a_{11}}\sum_{2\le j\le n} a_{1j}x_j\right)^2 - \left(\frac{1}{2a_{11}}\sum_{2\le j\le n} a_{1j}x_j\right)^2\right] + \sum_{2\le i,j\le n} a_{ij}x_ix_j.
\end{aligned}
$$

Setting

$$L_1(x_1,x_2,\dots,x_n) = x_1 + \frac{1}{2a_{11}}\sum_{2\le j\le n} a_{1j}x_j,$$

(which is the expression of a linear form on E), we obtain

$$q(x) = a_{11}L_1(x_1,x_2,\dots,x_n)^2 + \sum_{2\le i,j\le n} b_{ij}x_ix_j,$$

for certain $b_{ij}\in K$ (expressed in terms of the a_{ij}).
Note that the expression $\sum_{2\le i,j\le n} b_{ij}x_ix_j$ is a homogeneous polynomial of degree two in the $(n-1)$ variables $x_2,x_3,\dots,x_n$.

Second case: (If all a_{ii} are zero.)
Since q is not identically zero, at least one of the a_{ij} (with $i\neq j$) is nonzero. Up to permuting the coordinates $x_1,x_2,\dots,x_n$, we may assume that $a_{12}\neq 0$. Then for every $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$:

$$q(x) = a_{12}x_1x_2 + \sum_{\substack{1\le i,j\le n\\ (i,j)\neq(1,2)}} a_{ij}x_ix_j.$$

Performing the change of coordinates[2]

$$\begin{cases} x_1 = y_1 + y_2, \\ x_2 = y_1 - y_2, \\ x_i = y_i \quad (3\le i\le n), \end{cases}$$

We obtain:

$$q(x) = a_{12}y_1^2 - a_{12}y_2^2 + \sum_{1\le i\le j\le n} b_{ij}y_iy_j,$$

for certain $b_{ij}\in K$ (expressed in terms of the a_{ij}). We are thus reduced to the first case with the new coordinates $y_1,y_2,\dots,y_n$.

[2] A change of coordinates corresponds to a change of basis.

Conclusion: The above procedure (that is, the first case or the second case followed by the first) leads to writing $q(x)$ as a linear combination of the square of a linear form L_1 and homogeneous polynomials of degree $(n-1)$ in the linear forms $L_2, L_3, \ldots, L_n$ on E (where $L_1, L_2, \ldots, L_n$ are K-linearly independent). Iterating this process yields the desired reduction.

II.3 Real Quadratic Forms

Definition 3.1.7 A quadratic form q on $\mathbb{R}^n$ is:

- *positive* if $q(x) \geq 0$ for all x,
- *positive definite* if $q(x) > 0$ for all $x \neq 0$.

Theorem 3.1.3 — Cauchy-Schwarz inequality. If q is positive with polar form f, then

$$f(x,y)^2 \leq q(x)q(y).$$

Proof. Consider $q(x+ty)$ as a quadratic polynomial in t. Since it is nonnegative for all t, its discriminant is nonpositive. ■

Theorem 3.1.4 — Sylvester's Law of Inertia. Every real quadratic form admits a decomposition

$$q(x) = \sum_{i=1}^{p} (\varphi_i(x))^2 - \sum_{j=1}^{q} (\psi_j(x))^2,$$

and the pair (p,q) is independent of the chosen basis.

The integers (p,q) are called the *signature* of q.

$$\operatorname{rank}(q) = p+q.$$

Examples .

.

.

3.2 Norm Associated with an Inner Product

Definition 3.2.1
- A *real pre-Hilbert space* (or simply a pre-Hilbert space when there is no ambiguity about the field $\mathbb{R}$ used) is any real vector space equipped with an inner product.
- A *Euclidean space* is any finite-dimensional real pre-Hilbert space.

Let E be a real pre-Hilbert space and $\langle \cdot, \cdot \rangle$ its associated inner product. We define the mapping

$$\begin{aligned} \|\cdot\| : E &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto \|x\| := \sqrt{\langle x, x \rangle}. \end{aligned}$$

We will show later that $\|\cdot\|$ defines a norm on E, which justifies this notation. This norm $\|\cdot\|$ is called the *norm associated with the inner product* $\langle \cdot, \cdot \rangle$ on E.

Unless otherwise stated, the inner product of a pre-Hilbert space is denoted $\langle \cdot, \cdot \rangle$, and its associated norm is denoted $\|\cdot\|$.

In what follows, we present the fundamental properties satisfied by the mapping $\|\cdot\|$.

Proposition 5.1 (Cauchy–Schwarz Inequality). — Let E be a pre-Hilbert space. Then, for all $x, y \in E$, we have

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|. \tag{5.1}$$

Moreover, equality holds if and only if x and y are collinear.

4. Introduction to Hermitian Spaces

4.1 Introduction

Hermitian spaces are the natural adaptation of the tools of Euclidean geometry to the complex setting.

4.2 Preliminary Notions

4.2.1 Semilinear Maps and Sesquilinear Forms

Let E and F be two vector spaces over $\mathbb{C}$.

Definition 4.2.1 (Semilinear map from E to F). A mapping $f : E \longrightarrow F$ is called **semilinear** if it satisfies:

$$\forall x, x' \in E,\ \forall \lambda \in \mathbb{C}, \quad \begin{cases} f(x+x') = f(x) + f(x'), \\ f(\lambda x) = \overline{\lambda}\, f(x). \end{cases}$$

We denote by $\mathscr{L}^{1/2}(E,F)$ the set of semilinear maps from E to F.

When $F = \mathbb{C}$, one speaks of a **semilinear form** on E. A map f is a semilinear form if and only if f is a linear form.

■ **Example 4.1** ■

Definition 4.2.2 (Left sesquilinear form on E). A mapping $\varphi : E \times E \longrightarrow \mathbb{C}$ is called a **left sesquilinear form** if it satisfies:

- For all $x, x', y \in E$ and for all $\lambda \in \mathbb{C}$, $\quad \varphi(\lambda x + x', y) = \lambda \varphi(x,y) + \varphi(x', y)$.
- For all $x, y, y' \in E$ and for all $\mu \in \mathbb{C}$, $\quad \varphi(x, \mu y + y') = \overline{\mu}\, \varphi(x,y) + \varphi(x, y')$.

We denote by $\mathscr{L}^{3/2}(E)$ the set of left sesquilinear forms on E.

The previous conditions may also be expressed as:

- For every $y \in E$, the mapping $x \mapsto \varphi(x,y)$ is linear.
- For every $x \in E$, the mapping $y \mapsto \varphi(x,y)$ is semilinear.

■ **Example 4.2** ■

Definition 4.2.3 (Terminology for sesquilinear forms).

Let $\varphi \in \mathscr{L}^{3/2}(E)$. We say that φ is:

- **Hermitian** if

$$\forall x, y \in E, \quad \varphi(x,y) = \overline{\varphi(y,x)}.$$

- **Anti-Hermitian** if

$$\forall x, y \in E, \quad \varphi(x,y) = -\overline{\varphi(y,x)}.$$

- **Definite** if

$$\varphi(x,x) = 0 \iff x = 0.$$

- **Positive** if

$$\forall x \in E, \quad \varphi(x,x) \geq 0.$$

■ **Example 4.3**

■

.

4.2.2 Conjugate Matrix, Adjoint Matrix

The contribution of the complex structure, compared with the real structure, is the possibility of using complex conjugation:

$$\begin{aligned} c : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \overline{z}. \end{aligned}$$

It is an involutive automorphism of the field $\mathbb{C}$.

Definition 4.2.4 (Conjugate matrix).
The conjugate matrix of a matrix $M = (a_{i,j}) \in M_n(\mathbb{C})$ is defined by

$$\overline{M} = (\overline{a_{i,j}}).$$

■ **Example 4.4** ■

Proposition 4.2.1 (Algebraic properties of conjugation). The mapping

$$M \longmapsto \overline{M}$$

is a semilinear isomorphism of $M_n(\mathbb{C})$.

Moreover, for all $M, N \in M_n(\mathbb{C})$, we have:

1. $\operatorname{rank}(\overline{M}) = \operatorname{rank}(M)$ and $\det(\overline{M}) = \overline{\det(M)}$;
2. $\overline{\overline{M}} = M$ and $\overline{MN} = \overline{M}\,\overline{N}$;
3. if $M \in \mathrm{GL}_n(\mathbb{C})$, then $\overline{M} \in \mathrm{GL}_n(\mathbb{C})$ and

$$(\overline{M})^{-1} = \overline{M^{-1}}.$$

Since $\overline{\overline{M}} = M$, the mapping is involutive; hence it is bijective and equal to its own inverse.

We now introduce a notion adapted to sesquilinear computations.

Definition 4.2.5 (Adjoint matrix).

The adjoint matrix of $M \in M_n(\mathbb{C})$ is defined by

$$M^* := {}^t\overline{M}.$$

The mapping

$$\begin{aligned} M_n(\mathbb{C}) &\longrightarrow M_n(\mathbb{C}) \\ M &\longmapsto M^* \end{aligned}$$

is called the **adjunction**.

■ **Example 4.5** ■

Proposition 4.2.2 (Algebraic properties of adjunction).
The mapping $M \mapsto M^*$ is a semilinear isomorphism of $M_n(\mathbb{C})$.
For all $M, N \in M_n(\mathbb{C})$, we have:
1. $\operatorname{rank}(M^*) = \operatorname{rank}(M)$ and $\det(M^*) = \overline{\det(M)}$;
2. $(M^*)^* = M$ and $(MN)^* = N^*M^*$;
3. if $M \in \mathrm{GL}_n(\mathbb{C})$, then $M^* \in \mathrm{GL}_n(\mathbb{C})$ and

$$(M^*)^{-1} = (M^{-1})^*.$$

4.2.3 Normal Matrices

Definition 4.2.6 (Normal matrix).
A matrix $M \in M_n(\mathbb{C})$ is said to be **normal** if

$$MM^* = M^*M.$$

It will be shown later that every normal matrix is diagonalizable.

■ **Example 4.6** ■

Proposition 4.2.3 (Definition of a unitary matrix). Let $M \in M_n(\mathbb{C})$. The following conditions are equivalent:
1. $M^*M = I_n$;
2. $MM^* = I_n$;

3. $M \in \mathrm{GL}_n(\mathbb{C})$ and $M^{-1} = M^*$.

In this case, M is called a **unitary matrix** of order n.

Each of these conditions may also be written as

$$\forall i, j \in \{1, \ldots, n\}, \quad \sum_{k=1}^{n} a_{i,k}\overline{a_{j,k}} = \delta_{i,j}.$$

4.2.4 Hermitian Matrices

Definition 4.2.7 (Hermitian matrix).

A matrix $M \in M_n(\mathbb{C})$ is said to be **Hermitian** if

$$M^* = M.$$

In other words,

$$M = {}^t\overline{M}.$$

Equivalently, if $M = (a_{i,j})$, then

$$a_{i,j} = \overline{a_{j,i}} \quad \text{for all } i, j.$$

In particular, the diagonal entries of a Hermitian matrix are real numbers.

■ **Example 4.7 — Hermitian matrix.** ■

Proposition 12 (Basic properties).

Let $M, N \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{R}$.

1. If M and N are Hermitian, then $M + N$ is Hermitian.
2. If M is Hermitian, then λM is Hermitian.
3. If M is Hermitian, then iM is skew-Hermitian.

Definition 4.2.8 (Skew-Hermitian matrix). A matrix $M \in M_n(\mathbb{C})$ is said to be **skew-Hermitian** if

$$M^* = -M.$$

Equivalently,

$$a_{i,j} = -\overline{a_{j,i}} \quad \text{for all } i, j.$$

In particular, the diagonal entries of a skew-Hermitian matrix are purely imaginary.

■ **Example 4.8 — Skew-Hermitian matrix.** ■

Proposition 14 (Decomposition).
Every matrix $M \in M_n(\mathbb{C})$ can be uniquely written as

$$M = H + S,$$

where

$$H = \frac{1}{2}(M + M^*) \quad \text{and} \quad S = \frac{1}{2}(M - M^*),$$

with H Hermitian and S skew-Hermitian.

Proposition 15 (Spectral property).
If M is Hermitian and λ is an eigenvalue of M, then $\lambda \in \mathbb{R}$.

Proof. Let $x \neq 0$ such that $Mx = \lambda x$. Then

$$\langle Mx, x \rangle = \lambda \langle x, x \rangle.$$

Since M is Hermitian,

$$\langle Mx, x\rangle = \langle x, Mx\rangle = \overline{\lambda}\langle x, x\rangle.$$

Hence,

$$\lambda\langle x, x\rangle = \overline{\lambda}\langle x, x\rangle.$$

Because $\langle x, x\rangle > 0$, we obtain

$$\lambda = \overline{\lambda},$$

which implies that λ is real. $\square$

4.3 Hermitian Space

Let E be a complex vector space of finite dimension $n \geq 1$.

4.3.1 Hermitian Inner Product

Definition 4.3.1 — Hermitian inner product on E. A sesquilinear mapping

$$\varphi : E \times E \longrightarrow \mathbb{C}$$

is called a Hermitian inner product on E if it satisfies the following properties:

1. φ is positive, i.e. $\forall x \in E$, $\varphi(x, x) \geq 0$;
2. φ is definite, i.e. $\forall x \in E$,

$$\varphi(x, x) = 0 \Rightarrow x = 0.$$

The pair (E, φ) is then called a Hermitian space.

We will often write $\langle\cdot,\cdot\rangle$ instead of φ for the inner product of a Hermitian space, and simply write E instead of $(E, \langle\cdot,\cdot\rangle)$.

■ **Example 4.9 — Canonical inner product on $\mathbb{C}^n$.** ■

■ **Example 4.10 — Inner product on $M_n(\mathbb{C})$.** ■

R [Hermitian subspace] Let $(E, \langle\cdot,\cdot\rangle)$ be a Hermitian space and let $E' \subset E$ be a vector subspace.
The restriction of the inner product to E' defines a Hermitian inner product on E'.
We say that E' is a Hermitian subspace of E.

4.3.2 Expressions in a Basis

Let $(E, \langle\cdot,\cdot\rangle)$ be a Hermitian space of dimension $n \geq 1$.

Definition 4.3.2 — Gram matrix and Gram determinant. Let $x = (x_1, \ldots, x_n)$ be a family of vectors in E.

The Gram matrix of x is defined by:

$$\mathrm{Mat}(\langle\cdot,\cdot\rangle; x) = \begin{pmatrix} \langle x_1, x_1\rangle & \cdots & \langle x_1, x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \cdots & \langle x_n, x_n\rangle \end{pmatrix}.$$

Its determinant is called the Gram determinant and is denoted by

$$\mathrm{Gram}(x).$$

The Gram matrix is Hermitian since

$$\langle x_j, x_i\rangle = \overline{\langle x_i, x_j\rangle}.$$

R If $e = (e_1, \ldots, e_n)$ is a basis of E, the matrix

$$\Omega = \mathrm{Mat}(\langle\cdot,\cdot\rangle; e)$$

is called the matrix of the inner product in the basis e.
Its determinant is called the discriminant and is denoted by $\Delta(e)$.

If

$$x = \sum_{i=1}^{n} x_i e_i, \qquad y = \sum_{j=1}^{n} y_j e_j,$$

and if X, Y denote their coordinate column vectors, then:

$$\langle x, y\rangle = \sum_{i,j} \omega_{ij} x_i \overline{y_j} = {}^t\overline{X}\,\Omega Y.$$

Proposition 4.3.1 — Change of basis. Let e' be another basis of E and let P be the change-of-basis matrix from e to e'.

If Ω and Ω' are the matrices of the inner product in e and e' respectively, then:

$$\Omega' = {}^t\overline{P}\,\Omega P.$$

Corollary 4.3.2 Under the previous assumptions,

$$\Delta(e') = |\det P|^2 \Delta(e).$$

In particular, a family x is linearly independent if and only if

$$\mathrm{Gram}(x) \neq 0.$$