Chapter 2

Review of Probability Theory

Contents

1 Random experiment and event

1.1 Definitions

Definition 1 (Random experiment)

A random experiment (r.e.) is any experiment whose outcome is governed by chance.

Definition 2 (Sample space)

The set of all possible outcomes of a random experiment is called the sample space and is generally denoted by Ω.

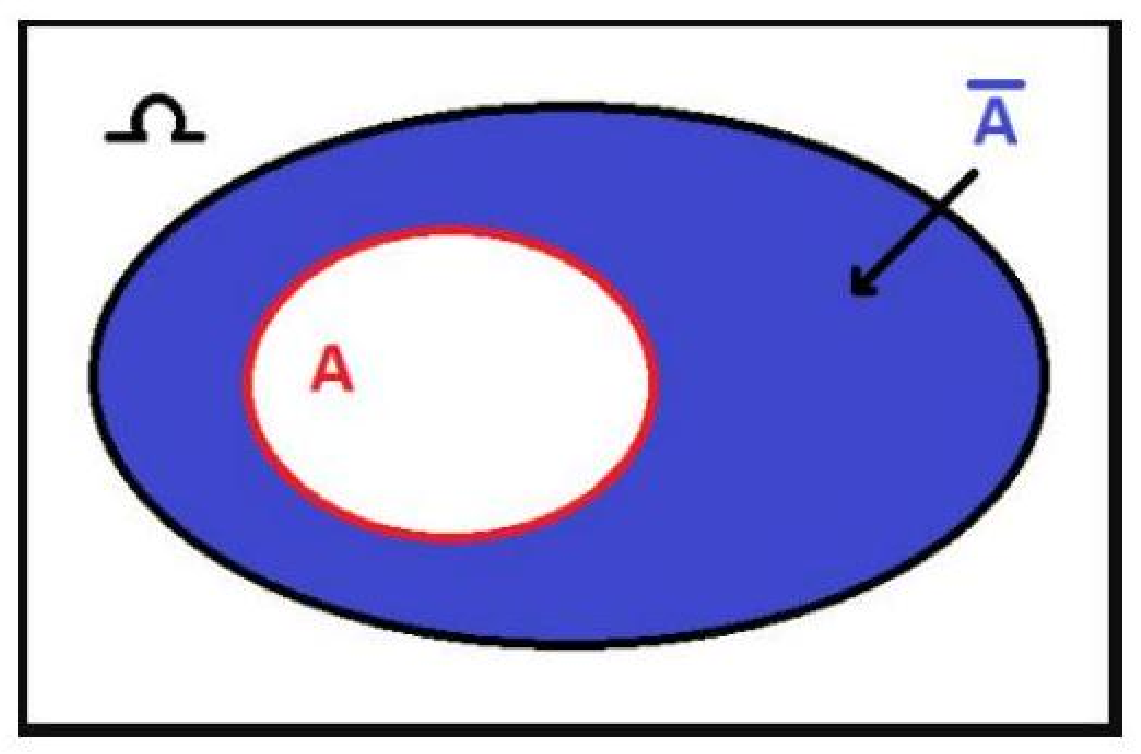
Definition 3 (Event)

An event of Ω is a subset of Ω.

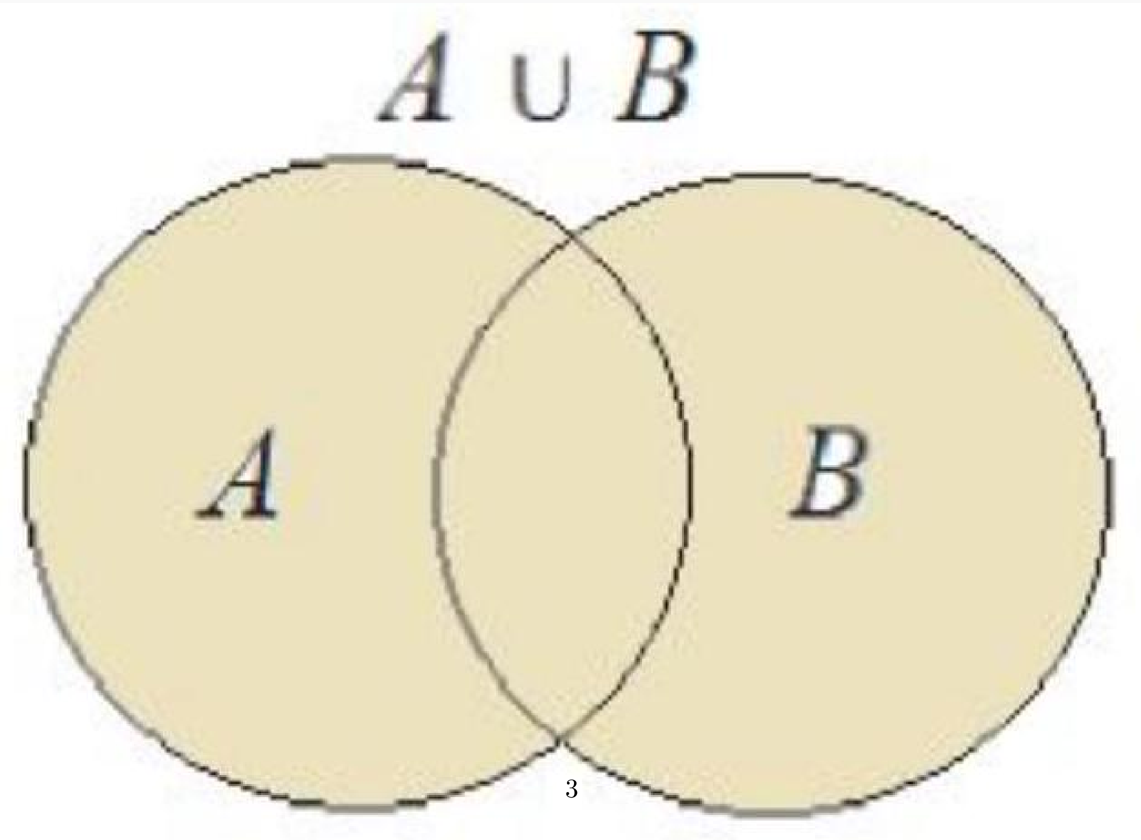
(1) An event is **certain** if it always occurs.

(2) An event is **impossible** if it never occurs.

(3) The complementary event of A is denoted by $\bar{A}$.



(4) $A \cup B$ occurs if A or B occurs.



(5) $A \cap B$ occurs if both A and B occur.

1.2 Examples

Example 2.1

The random experiment "rolling a six-faced numbered die".

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{2\}, \quad B = \{2, 4, 6\}, \quad \bar{B} = \{1, 3, 5\}, \quad C = \{1, 2, 3, 4, 5, 6\}$$

Event D is impossible.

$$B - A = \{4, 6\}, \quad A \cup B = \{2, 4, 6\}, \quad A \cap B = \{2\}$$

A and B are not incompatible.

Definition 4 (Classical probability)

$$P(A) = \frac{|A|}{|\Omega|}$$

2 General Concepts

Definition (Axiomatic probability)

A probability is a function $P : \Omega \to [0, 1]$ such that for every event $A \in \Omega$:

1. $P(A) \geq 0$,
2. $P(\Omega) = 1$,
3. If $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$.

Elementary properties

1. $P(\overline{A}) = 1 - P(A)$
2. $P(\varnothing) = 0$
3. If $A \subset B$, then $P(A) \leq P(B)$
4. $0 \leq P(A) \leq 1$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example (Complement)

Question: In a wheat field, the probability that a seed germinates is $P(A) = 0.9$. What is the probability that it does not germinate?

Answer: $P(\overline{A}) = 1 - P(A) = 0.1$ (10% chance).

Conditional Probability

For events A and B with $P(B) \neq 0$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Example (Conditional Probability)

Scenario: $A =$ "seed germinates", $B =$ "seed treated with growth regulator". Suppose $P(A \mid B) = 0.9$ for treated seeds and $P(A \mid B^c) = 0.7$ for untreated seeds.

Model Answer:

1. $P(A \mid B) = 0.9$ means **given** the seed was treated, the probability of germination is 90%.
2. Joint probability $P(A \cap B) = P(B)\, P(A \mid B)$.
3. If 40% seeds treated, $P(B) = 0.4$, $P(A \cap B) = 0.4 \times 0.9 = 0.36$.
4. Conditional probability separates treatment effect from population probability.

Law of Total Probability

If $B_1, \ldots, B_n$ form a partition of Ω, then for any event A,

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A \mid B_i)$$

Example (Total Probability)

Humidity levels B_1 low, B_2 medium, B_3 high; $P(A \mid B_i)$ known. Compute $P(A)$:

$$P(A) = 0.3 \cdot 0.1 + 0.4 \cdot 0.3 + 0.3 \cdot 0.6 = 0.33$$

Overall probability of pest detection is 33%.

Bayes' Theorem

For events A, B with $P(B) > 0$,

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$$

If $\{A_i\}$ is a partition,

$$P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{\sum_i P(A_i)P(B \mid A_i)}$$

Example (Bayes)

A=tomato plant infected (5%), B=yellow spots observed, $P(B \mid A) = 0.9$, $P(B) = 0.1$.

$$P(A \mid B) = \frac{0.05 \cdot 0.9}{0.1} = 0.45$$

Probability plant is infected given yellow spots is 45%.

Independence

Events A and B are independent iff $P(A \cap B) = P(A)P(B)$. Equivalently $P(A \mid B) = P(A)$ if $P(B) > 0$.

Example (Independence)

A=soil rich in nitrogen, B=sunlight>6h, $P(A) = 0.7$, $P(B) = 0.8$, independent. Then

$$P(A \cap B) = 0.7 \times 0.8 = 0.56$$

56% of locations have both high nitrogen and sufficient sunlight.