1.8 Exercise Solutions

Solution 1.1

Let us analyze each case separately to determine if $(\mathbb{R}^2, +, \cdot)$ forms a vector space over $\mathbb{R}$.

(a)

The internal operation (vector addition) is the standard one, which satisfies all the required axioms. However, the external operation (scalar multiplication) is defined as:

$$\lambda \cdot (a, b) = (\lambda a, b).$$

This structure is **not** a vector space, because it violates the following vector space axiom:

- Distributivity of scalar multiplication with respect to field addition:

$$(\lambda + \mu) \cdot (a, b) = ((\lambda + \mu)a, b) \neq \lambda \cdot (a, b) + \mu \cdot (a, b) = ((\lambda + \mu)a, 2b),$$

  unless $b = 0$.

(b)

The scalar multiplication is defined as:

$$\lambda \cdot (a, b) = (\lambda^2 a, \lambda^2 b).$$

This structure is **not** a vector space, because the scalar multiplication is not linear with respect to field addition:

$$(\lambda+\mu)\cdot(a, b) = ((\lambda+\mu)^2 a, (\lambda+\mu)^2 b) \neq \lambda\cdot(a, b)+\mu\cdot(a, b) = ((\lambda^2+\mu^2)a, (\lambda^2+\mu^2)b).$$

Thus, it is not a vector space.

(c)

The operations are defined as:

$$(a, b) + (c, d) = (a + c, b + d), \quad \lambda \cdot (a, b) = (\lambda a, \lambda b).$$

This is the standard vector space structure on $\mathbb{R}^2$. All vector space axioms are satisfied:

- Associativity and commutativity of addition.
- Existence of additive identity $(0, 0)$ and additive inverses $(-a, -b)$.
- Distributivity and compatibility of scalar multiplication.
- Scalar identity $1 \cdot (a, b) = (a, b)$ holds.

Thus, this is a vector space over $\mathbb{R}$.

Solution 1.2

Determine which of the given sets are vector subspaces.

1. E_1: **Subspace.** Closed under addition and scalar multiplication (solution set of a homogeneous linear equation).
2. E_2: **Not a subspace.** Does not contain the zero vector, since $0 + 0 + 0 \neq 5$.
3. E_3: **Subspace.** The conditions $x = 2y = 3z = 4t$ define a line through the origin in $\mathbb{R}^4$.
4. E_4: **Not a subspace.** Not closed under addition; for example, $(1, 0)+(0, 1) = (1, 1)$ is not on the unit circle.
5. E_5: **Not a subspace.** Contains the zero vector, but not closed under addition. For example, $(1, 1) \in E_5$ and $(2, 8) \in E_5$, but $(1 + 2, 1 + 8) = (3, 9) \notin E_5$, because $9 \neq 3^3 = 27$.
6. E_6: **Subspace.** Intersection of two planes through the origin (solution space of a homogeneous system).
7. E_7: **Not a subspace.** The union of two planes is not closed under addition (take vectors from different planes).
8. E_8: **Subspace.** The condition $P(0) = P(1)$ is linear and preserved under polynomial operations.

9. E_9: **Subspace.** Polynomials of degree ≤ 2 are closed under addition and scalar multiplication.

10. E_{10}: **Not a subspace.** The divisibility condition $P' \mid P$ is not preserved under addition; for example, X^2 and X^3 satisfy it, but their sum does not.

11. E_{11}: **Subspace.** Bounded functions form a subspace of $F(\mathbb{R}, \mathbb{R})$.

12. E_{12}: **Not a subspace.** Functions bounded below are not closed under scalar multiplication, since multiplying by -1 gives functions unbounded above.

13. E_{13}: **Subspace.** Solutions of the linear differential equation $f' + 3f = 0$ form a vector space.

14. E_{14}: **Subspace.** The integral condition $\int_a^b f(t)\,dt = 0$ is linear and preserved under addition and scalar multiplication.

Solution 1.3

1. To determine if $t = (2, 1, 0, -3)$ belongs to the span of $\{u, v, w\}$, we solve:

$$t = \alpha u + \beta v + \gamma w.$$

This gives the system:

$$\begin{cases} 2\alpha + \beta = 2, \\ 3\alpha - \beta + \gamma = 1, \\ \alpha + 2\beta + 3\gamma = 0, \\ 3\beta - \gamma = -3. \end{cases}$$

Solving: from the first equation, $\beta = 2 - 2\alpha$. Substituting into the fourth equation gives $\gamma = 9 - 6\alpha$. Plugging these into the third equation yields $\alpha = 1$, leading to $\beta = 0$ and $\gamma = 3$. These values satisfy all equations; therefore, t is in the span.

2. To find the equation of the plane G spanned by $(3, 1, -2)$ and $(1, -1, 4)$:

(a) **Parametric approach:** any vector in G can be written as

$$(x, y, z) = \lambda(3, 1, -2) + \mu(1, -1, 4) = (3\lambda + \mu, \lambda - \mu, -2\lambda + 4\mu).$$

(b) **Eliminate parameters:** from the first two components,

$$x = 3\lambda + \mu, \quad y = \lambda - \mu.$$

Solving for λ and μ,

$$\lambda = \frac{x+y}{4}, \quad \mu = \frac{x-3y}{4}.$$

(c) **Substitute into the third component:**

$$z = -2\lambda + 4\mu = -2\frac{x+y}{4} + 4\frac{x-3y}{4} = \frac{2x-14y}{4}.$$

Simplifying gives

$$2x - 14y - 4z = 0, \quad \text{or equivalently,} \quad x - 7y - 2z = 0.$$

Thus, G is the solution space of $x - 7y - 2z = 0$.

Solution 1.4

1. **Subspace verification:**
All three sets are subsets of the vector space $\mathbb{R}^{\mathbb{N}}$ (the space of all real sequences). We verify the subspace conditions:

- For E (convergent sequences):
  - Contains the zero sequence, which converges to 0.
  - If $(x_n), (y_n) \in E$ with limits L_x, L_y, then $(x_n + y_n)$ converges to $L_x + L_y$.
  - For $\alpha \in \mathbb{R}$, (αx_n) converges to αL_x.
- For F (sequences converging to 0):
  - Contains the zero sequence.
  - If $(x_n), (y_n) \in F$, then $(x_n + y_n)$ converges to $0 + 0 = 0$.
  - For $\alpha \in \mathbb{R}$, (αx_n) converges to $\alpha \cdot 0 = 0$.
- For G (constant sequences):
  - Contains the zero sequence.

- If $(x_n) = (c_1)$ and $(y_n) = (c_2)$ are constant, then $(x_n + y_n) = (c_1 + c_2)$ is constant.
- For $\alpha \in \mathbb{R}$, $(\alpha x_n) = (\alpha c_1)$ is constant.

2. **Direct sum decomposition** $E = F \oplus G$**:**
We must show:

- $E = F + G$; every convergent sequence can be written as the sum of a sequence converging to 0 and a constant sequence.
- $F \cap G = \{0\}$; the only constant sequence converging to 0 is the zero sequence.

Proof:
For any $(x_n) \in E$ with limit L, write:

$$x_n = (x_n - L) + L,$$

where $(x_n - L) \in F$ (converges to 0), and $L \in G$ (constant sequence).
If $(z_n) \in F \cap G$, then it is constant and converges to 0, so $z_n = 0$ for all n.
Thus, E is the direct sum of F and G.

Solution 1.5

1. Consider the linear combination:

$$\lambda_1(1, -2, 3) + \lambda_2(0, 1, -1) + \lambda_3(2, 3, 5) = (0, 0, 0).$$

This gives the system:

$$\begin{cases} \lambda_1 + 2\lambda_3 = 0, \\ -2\lambda_1 + \lambda_2 + 3\lambda_3 = 0, \\ 3\lambda_1 - \lambda_2 + 5\lambda_3 = 0. \end{cases}$$

The only solution is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, the family is linearly independent.
2. For the vectors $(3, -1, 4), (1, 0, -2), (5, -3, 10)$:

$$(5, -3, 10) = 2(3, -1, 4) - (1, 0, -2).$$

This non-trivial linear combination equals zero; therefore, the family is linearly dependent.
3. Any four vectors in $\mathbb{R}^3$ must be linearly dependent by dimension considerations. Thus, the family is linearly dependent.
4. For the vectors in $\mathbb{R}^4$:

$$\lambda_1(2, -1, 0, 3) + \lambda_2(0, 1, -2, -1) + \lambda_3(5, -2, 4, -3) = (0, 0, 0, 0).$$

Solving the resulting system shows that the only solution is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, the family is linearly independent.
5. In $\mathbb{R}_2[X]$, consider:

$$a \cdot 1 + bX + c(1 + X^2) = 0.$$

This implies $a + c = 0$, $b = 0$, and $c = 0$. The only solution is $a = b = c = 0$; so the family is linearly independent.
6. In $\mathbb{R}_3[X]$, the polynomials $1, X^2 - X, X^3 + 2X$ have distinct degrees. Any linear combination:

$$a \cdot 1 + b(X^2 - X) + c(X^3 + 2X) = 0$$

must have $a = b = c = 0$, by comparing coefficients. Thus, the family is linearly independent.

Solution 1.6

We will prove that $G = H$ under the given conditions.
Given:

1. $F + G = F + H$;
2. $F \cap G = F \cap H$;
3. $G \subset H$.

To show: $G = H$.
Proof:
Since $G \subset H$, by condition (3), we only need to prove that $H \subset G$.
Let $h \in H$ be arbitrary. From condition (1), since $h \in H \subset F + H = F + G$, there

exist $f \in F$ and $g \in G$ such that

$$h = f + g.$$

Now, since $G \subset H$ and $h \in H$, we have $f = h - g \in H$ (because H is a subspace). Thus,

$$f \in F \cap H.$$

By condition (2), $F \cap H = F \cap G$, so $f \in F \cap G$. This implies $f \in G$ (since $F \cap G \subset G$).
Therefore,

$$h = f + g \in G + G = G,$$

because G is a subspace (closed under addition).
Since $h \in H$ was arbitrary, we conclude that $H \subset G$.
Together with condition (3), $G \subset H$, this proves that $G = H$.

Solution 1.7

Part 1: Proving that $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.
To show that the vectors $v_1 = (0, 1, 1)$, $v_2 = (1, 0, 1)$, and $v_3 = (1, 1, 0)$ form a basis for $\mathbb{R}^3$, it is sufficient to verify either linear independence or that they span $\mathbb{R}^3$, since we have three vectors in a three-dimensional space.
Method: Checking linear independence.
Consider the linear combination:

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0,$$

which gives the system:

$$\begin{cases} 0\alpha + 1\beta + 1\gamma = 0, \\ 1\alpha + 0\beta + 1\gamma = 0, \\ 1\alpha + 1\beta + 0\gamma = 0. \end{cases}$$

Solving this system:
From the first equation: $\beta + \gamma = 0 \Rightarrow \beta = -\gamma$;
from the second equation: $\alpha + \gamma = 0 \Rightarrow \alpha = -\gamma$;
substitute into the third equation: $(-\gamma) + (-\gamma) = -2\gamma = 0 \Rightarrow \gamma = 0$.

Thus, $\alpha = \beta = \gamma = 0$. The only solution is the trivial one, proving that the vectors are linearly independent.
Since we have three linearly independent vectors in $\mathbb{R}^3$, they form a basis.
Part 2: Finding the components of $w = (1, 1, 1)$.
We need to find scalars x, y, z such that:

$$w = xv_1 + yv_2 + zv_3,$$

which gives the system:

$$\begin{cases} 0x + 1y + 1z = 1, \\ 1x + 0y + 1z = 1, \\ 1x + 1y + 0z = 1. \end{cases}$$

Solving:
From the first equation: $y + z = 1$;
from the second equation: $x + z = 1$;
from the third equation: $x + y = 1$.
Subtract (2) from (1): $y - x = 0 \Rightarrow y = x$;
substitute into (3): $x + x = 1 \Rightarrow x = \frac{1}{2}$;
thus, $y = \frac{1}{2}$;
from (2): $\frac{1}{2} + z = 1 \Rightarrow z = \frac{1}{2}$.
Therefore, the coordinates of w in the basis $\{v_1, v_2, v_3\}$ are:

$$w = \frac{1}{2}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3.$$

Solution 1.8

1. Showing that E is a subspace.

- **Contains zero vector:** $(0, 0, 0)$ satisfies both equations $x + y - z = 0$ and $x - y - z = 0$.
- **Closed under addition:** For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in E$:

$$(x_1+x_2)+(y_1+y_2)-(z_1+z_2) = 0, \quad (x_1+x_2)-(y_1+y_2)-(z_1+z_2) = 0.$$

- **Closed under scalar multiplication:** For $\alpha \in \mathbb{R}$ and $(x, y, z) \in E$:

$$\alpha x + \alpha y - \alpha z = 0, \quad \alpha x - \alpha y - \alpha z = 0.$$

Thus, E is a subspace.

2. Basis for E: Solve the system:

$$\begin{cases} x + y - z = 0, \\ x - y - z = 0. \end{cases}$$

Adding equations: $2x - 2z = 0 \Rightarrow x = z$. Substituting: $y = 0$. Therefore,

$$E = \{(z, 0, z) \mid z \in \mathbb{R}\}.$$

A basis is $\{(1, 0, 1)\}$, which is linearly independent and generates E.

3. Basis for F: The plane $x + y - 2z = 0$ has dimension 2. Verify:

$$b = (1, 1, 1) \in F : \; 1 + 1 - 2(1) = 0;$$

$$c = (0, 2, 1) \in F : \; 0 + 2 - 2(1) = 0.$$

Linear independence: $\alpha(1, 1, 1) + \beta(0, 2, 1) = 0$ has only $\alpha = \beta = 0$. Hence, $\{b, c\}$ is a basis of F.

4. Linear independence of $\{a, b, c\}$: Solve $\alpha(1, 0, 1) + \beta(1, 1, 1) + \gamma(0, 2, 1) = 0$:

$$\begin{cases} \alpha + \beta = 0, \\ \beta + 2\gamma = 0, \\ \alpha + \beta + \gamma = 0. \end{cases}$$

The only solution is $\alpha = \beta = \gamma = 0$. So the family is linearly independent.

5. Direct sum $E \oplus F = \mathbb{R}^3$:

- $E \cap F = \{0\}$: The only solution to $x = z, \; y = 0, \; x + y - 2z = 0$ is $(0, 0, 0)$.
- $\dim E + \dim F = 1 + 2 = 3 = \dim \mathbb{R}^3$.

Therefore, $E \oplus F = \mathbb{R}^3$.

6. Expressing $u = (x, y, z)$ in the basis $\{a, b, c\}$: Solve

$$\alpha(1,0,1) + \beta(1,1,1) + \gamma(0,2,1) = (x, y, z),$$

which gives the system:

$$\begin{cases} \alpha + \beta = x, \\ \beta + 2\gamma = y, \\ \alpha + \beta + \gamma = z. \end{cases}$$

Solution:

$$\gamma = z - x, \quad \beta = y - 2(z - x), \quad \alpha = x - (y - 2(z - x)).$$

Thus,

$$u = (3x - y - 2z)a + (y - 2z + 2x)b + (z - x)c.$$

Solution 1.9

1. Is (a, b, c, d) a basis of $\mathbb{R}^3$?
No. $\mathbb{R}^3$ has dimension 3, so any basis must consist of exactly three linearly independent vectors. Here we have four vectors, which are necessarily linearly dependent; therefore, they cannot form a basis.

2. Show that (a, b) is a basis of E:
Step 1: Check linear independence. Solve $\alpha(2, -1, -1) + \beta(-1, 2, 3) = (0, 0, 0)$:

$$\begin{cases} 2\alpha - \beta = 0, \\ -\alpha + 2\beta = 0, \\ -\alpha + 3\beta = 0. \end{cases}$$

The only solution is $\alpha = \beta = 0$. Hence, a and b are linearly independent.
Step 2: Express c and d as linear combinations of a and b:

$$c = 2a + 3b = (1, 4, 7), \quad d = a + b = (1, 1, 2).$$

Hence, $E = \text{span}(a, b)$, and (a, b) is a basis.
3. Determine equations characterizing E:

For $(x, y, z) \in E:$

$$(x, y, z) = \alpha(2, -1, -1) + \beta(-1, 2, 3),$$

which gives

$$x = 2\alpha - \beta, \quad y = -\alpha + 2\beta, \quad z = -\alpha + 3\beta.$$

Solving for α and $\beta:$

$$\alpha = \frac{2x + y}{3}, \quad \beta = \frac{x + 2y}{3}.$$

Substitute into $z:$

$$z = \frac{x + 5y}{3} \implies x + 5y - 3z = 0.$$

Thus,

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + 5y - 3z = 0\}.$$

4. Complete to a basis of $\mathbb{R}^3$:

Choose a vector not in E, e.g., $v = (1, 0, 0):$

$$1 + 5 \cdot 0 - 3 \cdot 0 = 1 \neq 0.$$

Check linear independence of $(a, b, v):$

$$\det \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ -1 & 3 & 0 \end{pmatrix} = -1 \neq 0.$$

Hence, (a, b, v) is a basis of $\mathbb{R}^3$. One possible completion: $(a, b, (1, 0, 0))$.