1.6 Exercises

Exercise 1.1: Vector Space Verification

Determine whether $\mathbb{R}^2$, equipped with the following operations, forms an $\mathbb{R}$-vector space.

(a) For all $(a, b), (c, d) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$(a, b) + (c, d) = (a + c, b + d), \quad \lambda \cdot (a, b) = (\lambda a, b).$$

(b) For all $(a, b), (c, d) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$(a, b) + (c, d) = (a + c, b + d), \quad \lambda \cdot (a, b) = (\lambda^2 a, \lambda^2 b).$$

(c) For all $(a, b), (c, d) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$(a, b) + (c, d) = (a + c, b + d), \quad \lambda \cdot (a, b) = (\lambda a, \lambda b).$$

Exercise 1.2: Subspaces Verification

Among the following sets, determine which are, or are not, vector subspaces.

1. $E_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0\}$.
2. $E_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 5\}$.
3. $E_3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x = 2y = 3z = 4t\}$.
4. $E_4 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
5. $E_5 = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$.
6. $E_6 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\} \setminus \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$.
7. $E_7 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$.
8. $E_8 = \{P \in \mathbb{R}[X] \mid P(0) = P(1)\}$.
9. $E_9 = \{P \in \mathbb{R}[X] \mid \deg(P) \leq 2\}$.

10. $E_{10} = \{P \in \mathbb{R}[X] \mid P'(X) \text{ divides } P(X)\}$.

11. $E_{11} = \{f \in F(\mathbb{R}, \mathbb{R}) \mid f \text{ is bounded}\}$.

12. $E_{12} = \{f \in F(\mathbb{R}, \mathbb{R}) \mid f \text{ is bounded below}\}$.

13. $E_{13} = \{f \in C^1(\mathbb{R}, \mathbb{R}) \mid f' + 3f = 0\}$.

14. $E_{14} = \{f \in C^1([a, b], \mathbb{R}) \mid \int_a^b f(t)\, dt = 0\}$.

Exercise 1.3

1. Let the vector $t = (2, 1, 0, -3)$. Does this vector belong to the subspace of $\mathbb{R}^4$ spanned by the vectors:

$$u = (2, 3, 1, 0), \quad v = (1, -1, 2, 3), \quad w = (0, 1, 3, -1)?$$

2. Let G be the subspace of $\mathbb{R}^3$ spanned by the vectors $(3, 1, -2)$ and $(1, -1, 4)$. Prove that there exist real numbers a, b, c such that

$$G = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\}.$$

Exercise 1.4

Let E be the set of all convergent real sequences, F the set of all real sequences converging to zero, and G the set of all constant real sequences.

1. Prove that E, F, and G are subspaces of $\mathbb{R}^{\mathbb{N}}$.

2. Prove that $E = F \oplus G$.

Exercise 1.5

Are the following families linearly independent?

1. $\{(1, -2, 3),\ (0, 1, -1),\ (2, 3, 5)\}$ in $\mathbb{R}^3$.

2. $\{(3, -1, 4),\ (1, 0, -2),\ (5, -3, 10)\}$ in $\mathbb{R}^3$.

3. $\{(1, 0, 0),\ (0, 2, 0),\ (1, 2, 3),\ (2, -1, -1)\}$ in $\mathbb{R}^3$.

4. $\{(2, -1, 0, 3),\ (0, 1, -2, -1),\ (5, -2, 4, -3)\}$ in $\mathbb{R}^4$.

5. $\{1,\ X,\ 1+X^2\}$ in $\mathbb{R}_2[X]$.

6. $\{1,\ X^2-X,\ X^3+2X\}$ in $\mathbb{R}_3[X]$.

Exercise 1.6

Let F, G, and H be three subspaces of a $\mathbb{K}$-vector space E such that

$$F+G=F+H, \quad F\cap G=F\cap H, \quad G\subseteq H.$$

Prove that $G=H$.

Exercise 1.7

Show that the vectors

$$v_1=(0,1,1), \quad v_2=(1,0,1), \quad v_3=(1,1,0)$$

form a basis of $\mathbb{R}^3$.
Find the components of the vector

$$w=(1,1,1)$$

in this basis (v_1, v_2, v_3).

Exercise 1.8: Subspaces and Bases in $\mathbb{R}^3$

Let

$$E=\{(x,y,z)\in\mathbb{R}^3 \mid x+y-z=0 \text{ and } x-y-z=0\},$$

$$F=\{(x,y,z)\in\mathbb{R}^3 \mid x+y-2z=0\}.$$

be two subsets of $\mathbb{R}^3$. Assume that F is a subspace of $\mathbb{R}^3$. Let

$$a=(1,0,1), \quad b=(1,1,1), \quad c=(0,2,1).$$

1. Show that E is a subspace of $\mathbb{R}^3$.

2. Determine a generating family of E and show that this family is a basis.

3. Show that $\{b, c\}$ is a basis of F.

4. Show that $\{a, b, c\}$ is a linearly independent family in $\mathbb{R}^3$.

5. Prove that $E \oplus F = \mathbb{R}^3$.

6. Let $u = (x, y, z)$. Express u in the basis $\{a, b, c\}$.

Exercise 1.9: Spanning and Completing Bases in $\mathbb{R}^3$

Let

$$E = \text{span}(a, b, c, d)$$

be a subspace of $\mathbb{R}^3$, where

$$a = (2, -1, -1), \quad b = (-1, 2, 3), \quad c = (1, 4, 7), \quad d = (1, 1, 2).$$

1. Is (a, b, c, d) a basis of $\mathbb{R}^3$?

2. Show that (a, b) is a basis of E.

3. Determine one or more equations characterizing E.

4. Complete a basis of E to form a basis of $\mathbb{R}^3$.

1.7 Additional exercises

Exercise 1.10: Subspaces and Direct Sum in $\mathbb{R}^4$

We consider the subset

$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y = 0 \text{ and } x + z = 0\}.$$

1. Show that F is a subspace of $\mathbb{R}^4$ and provide a basis for F.

2. Complete the basis found in part (1) to form a basis of $\mathbb{R}^4$.

3. Let

$$u_1 = (1, 1, 1, 1), \quad u_2 = (1, 2, 3, 4), \quad u_3 = (-1, 0, -1, 0).$$

Show whether the family $\{u_1, u_2, u_3\}$ is linearly independent.

4. Let G be the vector space spanned by u_1, u_2, u_3. Determine $\dim G$.

5. Provide a basis for $F \cap G$. Deduce that $F + G = \mathbb{R}^4$.

6. Can every vector in $\mathbb{R}^4$ be written uniquely as the sum of a vector from F and a vector from G?

Exercise 1.11: Subspaces and Bases in $\mathbb{R}^4$

Let

$$E = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y + z - t = 0,\ x - 2y + 2z + t = 0,\ x - y + z = 0\}$$

and assume that E is a vector space. Let

$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x + 6y + 7z - t = 0\}.$$

Let

$$a = (2, 1, -1, 2), \quad b = (1, 1, -1, 1), \quad c = (-1, -2, 3, 7), \quad d = (4, 4, -5, -3)$$

be vectors in $\mathbb{R}^4$.

1. (a) Determine a basis for E and deduce its dimension.

   (b) Complete this basis to form a basis of $\mathbb{R}^4$.

2. (a) Show that F is a subspace of $\mathbb{R}^4$.

   (b) Determine a basis for F.

   (c) Determine whether $E + F = \mathbb{R}^4$.

3. (a) Show that $F = \text{Span}\{b, c, d\}$.

   (b) Let $u = (x, y, z, t) \in F$. Express u as a linear combination of b, c, and d.

Exercise 1.12: Subspaces, Bases, and Direct Sum in $\mathbb{R}^3$

Let

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0 \text{ and } x - y - z = 0\}$$

and

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - 2z = 0\}$$

be two subsets of $\mathbb{R}^3$. Assume that F is a subspace of $\mathbb{R}^3$. Let

$$a = (1, 0, 1), \quad b = (1, 1, 1), \quad c = (0, 2, 1).$$

1. Show that E is a subspace of $\mathbb{R}^3$.
2. Determine a generating family for E and show that this family forms a basis.
3. Show that $\{b, c\}$ is a basis for F.
4. Show that $\{a, b, c\}$ is a linearly independent set in $\mathbb{R}^3$.
5. Prove that $E + F = \mathbb{R}^3$.
6. Let $u = (x, y, z)$. Express u in the basis $\{a, b, c\}$.

Exercise 1.13: Bases and Subspaces in $\mathbb{R}^3$

Let

$$u_1 = (1, -1, 2), \quad u_2 = (1, 1, -1), \quad u_3 = (-1, -5, -7), \quad E = \text{Span}\{u_1, u_2, u_3\}.$$

Let

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

1. Provide a basis for E.
2. Show that F is a subspace of $\mathbb{R}^3$.
3. Provide a basis for F.
4. Provide a basis for $E \cap F$.