Chapter 1

Vector spaces

In this course, the field $(\mathbb{K}, +, \cdot)$ denotes $\mathbb{R}$, $\mathbb{C}$, or any commutative field.

1.1 Definitions

Definition 1.1: Vector space

A vector space over a field $\mathbb{K}$ is a nonempty set E equipped with two operations:

1. **Vector addition (internal operation):** $E \times E \to E$, denoted by $(u, v) \mapsto u + v$.
2. **Scalar multiplication (external operation):** $\mathbb{K} \times E \to E$, denoted by $(\lambda, u) \mapsto \lambda u$.

These operations must satisfy the following axioms for all $u, v \in E$, and $\lambda, \mu \in \mathbb{K}$:

1. $(E, +)$ is an abelian group.
2. Scalar multiplication satisfies the following properties:
    (a) Distributivity over vector addition: $\lambda(u + v) = \lambda u + \lambda v$.
    (b) Distributivity over scalar addition: $(\lambda + \mu)u = \lambda u + \mu u$.
    (c) Compatibility of scalar multiplication: $\lambda(\mu u) = (\lambda\mu)u$.
    (d) Identity element of scalar multiplication: $1_{\mathbb{K}} u = u$, where $1_{\mathbb{K}}$ is the multiplicative identity in $\mathbb{K}$.

Rules of Calculation

The following properties hold in any vector space E over a field $\mathbb{K}$:

1. $\lambda \cdot x = 0_E \quad \Rightarrow \quad \lambda = 0_{\mathbb{K}},$ or $x = 0_E$.
2. For all $x \in E$ and all $\lambda \in \mathbb{K}$:
$$-(\lambda \cdot x) = (-\lambda) \cdot x = \lambda \cdot (-x).$$
3. For all $x \in E \setminus \{0_E\}$, and all $\lambda, \mu \in \mathbb{K}$:
$$\lambda \cdot x = \mu \cdot x \quad \Rightarrow \quad \lambda = \mu.$$
4. For all $x \in E$, and all $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$:
$$\sum_{k=1}^{n} (\lambda_k \cdot x) = \left(\sum_{k=1}^{n} \lambda_k \right) \cdot x.$$
5. For all $x_1, \ldots, x_n \in E$, and all $\lambda \in \mathbb{K}$:
$$\sum_{k=1}^{n} (\lambda \cdot x_k) = \lambda \cdot \left(\sum_{k=1}^{n} x_k \right).$$

Example 1.1: The Vector Space $\mathbb{R}^n$

The set $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ is a vector space over the field $\mathbb{R}$, equipped with the following operations:

- **Vector addition:** For $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, define:
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n).$$
- **Scalar multiplication:** For $\lambda \in \mathbb{R}$ and $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbb{R}^n$, define:
$$\lambda \cdot \mathbf{u} = (\lambda u_1, \ldots, \lambda u_n).$$

In particular:

- $\mathbb{R}^2$ is the 2-dimensional real vector space (the plane).
- $\mathbb{R}^3$ is the 3-dimensional real vector space.

All the vector space axioms (listed in Definition 1.1) are satisfied under these operations, so $\mathbb{R}^n$ is indeed a vector space over $\mathbb{R}$.

Example 1.2: Important Examples of Vector Spaces

Here are several important examples of vector spaces over common fields:

- **The Vector Space $\mathbb{C}^n$:** The set of all n-tuples of complex numbers:

  $$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) \mid z_i \in \mathbb{C}\}.$$

  It is a vector space over the field $\mathbb{C}$, with standard component-wise addition and scalar multiplication.

- **The Vector Space of Polynomials $\mathbb{R}[x]$:** The set of all real-coefficient polynomials:

  $$\mathbb{R}[x] = \left\{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R},\ n \in \mathbb{N}\right\}.$$

  It is a vector space over $\mathbb{R}$, with usual polynomial addition and scalar multiplication.

- **The Vector Space of Continuous Functions $\mathcal{C}([a,b])$:** Let $\mathcal{C}([a,b])$ denote the set of all real-valued continuous functions defined on the closed interval $[a,b]$. Then:

  $$\mathcal{C}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

  It is a vector space over $\mathbb{R}$, with pointwise addition and scalar multiplication defined by:

  $$(f+g)(x) = f(x) + g(x),$$
  $$(\lambda f)(x) = \lambda \cdot f(x),$$

for all $f, g \in \mathcal{C}([a,b])$, $\lambda \in \mathbb{R}$, and $x \in [a,b]$.

Definition 1.2: Linear Combination

Let E be a vector space over a field $\mathbb{K}$, and let $x_1, x_2, \ldots, x_n \in E$ be vectors. A vector $x \in E$ is called a *linear combination* of $x_1, x_2, \ldots, x_n$ if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$ such that:

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n.$$

In this case, we say that x is expressed as a linear combination of the vectors $x_1, x_2, \ldots, x_n$ with coefficients $\lambda_1, \ldots, \lambda_n$.

Example 1.3: Examples of Linear Combinations

Here are some examples of linear combinations in different vector spaces:

1. **In $\mathbb{R}^3$:**

   Let $\mathbf{x}_1 = (1, 0, 2)$, $\mathbf{x}_2 = (0, 1, -1)$. Then the vector

$$\begin{aligned}
\mathbf{x} &= 3\mathbf{x}_1 - 2\mathbf{x}_2 \\
&= 3(1, 0, 2) - 2(0, 1, -1) \\
&= (3, 0, 6) + (0, -2, 2) \\
&= (3, -2, 8)
\end{aligned}$$

   is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ with coefficients 3 and -2.

2. **In the space of polynomials $\mathbb{R}[x]$:**

   Let $p_1(x) = 1 + x$, $p_2(x) = x^2$. Then:

$$\begin{aligned}
p(x) &= 2p_1(x) - 5p_2(x) \\
&= 2(1 + x) - 5x^2 \\
&= 2 + 2x - 5x^2
\end{aligned}$$

   is a linear combination of p_1 and p_2 in $\mathbb{R}[x]$.

3. **In the space of continuous functions $\mathcal{C}([0, 1])$:**

Let $f_1(x) = x$, $f_2(x) = \sin(x)$. Then the function

$$f(x) = 4f_1(x) + \pi f_2(x) = 4x + \pi \sin(x)$$

is a linear combination of f_1 and f_2 in $\mathcal{C}([0, 1])$.

1.2 Subspaces

Definition 1.3: Subspace of a Vector Space

Let E be a vector space over a field $\mathbb{K}$. A subset $F \subseteq E$ is called a *vector subspace* of E (or simply a *subspace*) if it satisfies the following conditions:

1. **Non-empty:** $F \neq \emptyset$.
2. **Closed under vector addition:** For all $u, v \in F$, we have $u + v \in F$.
3. **Closed under scalar multiplication:** For all $\lambda \in \mathbb{K}$ and all $u \in F$, we have $\lambda u \in F$.

Proposition 1.1: Equivalent Characterization of Subspaces

Let E be a vector space over a field $\mathbb{K}$ and $F \subseteq E$ a subset. Then F is a subspace of E if and only if:

1. $F \neq \emptyset$,
2. For all $\lambda, \mu \in \mathbb{K}$ and all $u, v \in F$, we have $\lambda u + \mu v \in F$.

Remark 1.1: Zero Vector in Subspaces

Every subspace F contains the zero vector $\mathbf{0}_E$ of E.

Example 1.4: Important Examples of Subspaces

Important examples of subspaces:

- **Trivial subspaces:**

  - The zero subspace $\{\mathbf{0}_E\}$.
  - The entire space E itself.

- **In $\mathbb{R}^3$:**

  The set
  $$F = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - z = 0\}$$
  is a subspace of $\mathbb{R}^3$, while the set
  $$G = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + 3z = 1\}$$
  is not a subspace (it fails to contain $\mathbf{0}$).

- **Polynomial subspaces:**

  For any field $\mathbb{K}$ and integer $n \geq 0$, the set
  $$\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] \mid \deg(P) \leq n\}$$
  is a subspace of the polynomial space $\mathbb{K}[X]$.

- **Function spaces:**

  The set of all continuous functions $f \in \mathcal{C}([a, b])$ satisfying $f(a) = f(b) = 0$ forms a subspace of $\mathcal{C}([a, b])$.

Proposition 1.2: Properties of Subspaces

Let F be a subspace of a $\mathbb{K}$-vector space E. Then the following properties hold:

1. Any linear combination of vectors in F remains in F.
2. The intersection of any collection of subspaces of E is itself a subspace of E.
3. The union of two subspaces $F \cup G$ is a subspace if and only if $F \subseteq G$ or $G \subseteq F$.

Example 1.5: Union of Subspaces

Let $E_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ and $E_2 = \{(0, y) \mid y \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^2$. The union $E_1 \cup E_2$ is not a vector space.

Proof 1.1

To verify that $E_1 \cup E_2$ is not a vector space, consider the vectors:

- $\mathbf{u} = (1, 0) \in E_1$.
- $\mathbf{v} = (0, 1) \in E_2$.

Their sum $\mathbf{u} + \mathbf{v} = (1, 1)$ does not belong to $E_1 \cup E_2$, because:

- $(1, 1) \notin E_1$ (second component is not zero).
- $(1, 1) \notin E_2$ (first component is not zero).

Therefore, $E_1 \cup E_2$ is not closed under vector addition, and hence not a subspace.

Definition 1.4: Sum of Subspaces

Let E be a vector space over a field $\mathbb{K}$, and let F and G be two subspaces of E. The *sum* of F and G, denoted $F + G$, is defined as:

$$F + G = \{u + v \mid u \in F,\ v \in G\}.$$

Proposition 1.3: Properties of the Sum

Let F and G be subspaces of a vector space E over $\mathbb{K}$. The sum $F + G$ has the following properties:

1. $F + G$ is a vector subspace of E.
2. $F + G$ is the smallest subspace of E containing both F and G.
3. $F \cup G \subseteq F + G$.

Example 1.6: Sum and Intersection of Subspaces in $\mathbb{R}^3$

In $\mathbb{R}^3$, consider the following subspaces:

- $F = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ (the xy-plane).
- $G = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ (the yz-plane).

Then their sum is:

$$F + G = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3,$$

while their intersection is:

$$F \cap G = \{(0, y, 0) \mid y \in \mathbb{R}\} \quad \text{(the } y\text{-axis)}.$$

Remark 1.2: Properties of Subspace Sum

Let F, G, and H be three vector subspaces of E. The following properties hold:

1. **Algebraic properties:**
   - $F + G = G + F$ (Commutativity).
   - $F + (G + H) = (F + G) + H$ (Associativity).
   - $F + \{\mathbf{0}_E\} = F$ (Identity element).
   - $F + E = E$.
   - $F + F = F$.
2. **Non-invertibility:** If $H = F + G$, the expression $F = H - G$ does not make sense. Subtraction of subspaces is not defined in vector spaces.
3. **Non-cancellation property:** If $F + G = F + H$, we cannot immediately conclude that $G = H$.

Example 1.7: Non-Cancellation Property of Subspace Sum

In $\mathbb{R}^2$, take:

- $F = \{(x, 0) \mid x \in \mathbb{R}\}$ (x-axis),
- $G = \{(0, y) \mid y \in \mathbb{R}\}$ (y-axis),

- $H = \{(y, y) \mid y \in \mathbb{R}\}$ (diagonal line).

Then $F + G = F + H = \mathbb{R}^2$, but clearly $G \neq H$.

Definition 1.5: Direct Sum of Subspaces

Let E_1 and E_2 be two vector subspaces of a vector space E over $\mathbb{K}$. We say that E_1 and E_2 are in *direct sum* in E (or that E_1 is *complementary* to E_2 in E) if one of the following equivalent conditions is satisfied:

1. **Trivial intersection and spanning:** $E_1 \cap E_2 = \{\mathbf{0}_E\}$ and $E_1 + E_2 = E$.
2. **Unique decomposition:** For every vector $w \in E$, there exists a unique pair $(u, v) \in E_1 \times E_2$ such that $w = u + v$.

In this case, we write $E = E_1 \oplus E_2$.

Remark 1.3

The two conditions are equivalent because:

1. Condition (1) ensures the decomposition exists ($E_1 + E_2 = E$) and is unique (since $E_1 \cap E_2 = \{0_E\}$ prevents multiple representations).
2. Condition (2) implies that the intersection must be trivial, otherwise some vectors would have multiple decompositions.

Example 1.8: Direct Sum Decomposition of $\mathbb{R}^3$

Consider the following subspaces of $\mathbb{R}^3$:

$$E_1 = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle| x, z \in \mathbb{R} \right\} \quad \text{(the } xz\text{-plane).}$$

$$E_2 = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \middle| y \in \mathbb{R} \right\} \quad \text{(the } y\text{-axis).}$$

1. **Intersection Verification:**

$$E_1 \cap E_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The only vector common to both subspaces is the zero vector.

2. **Sum Verification:**

$$E_1 + E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\} = \mathbb{R}^3.$$

Any vector in $\mathbb{R}^3$ can be expressed as a sum of vectors from E_1 and E_2.

3. **Unique Decomposition:** For any $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, there exists a unique decomposition:

$$\mathbf{v} = \underbrace{\begin{pmatrix} a \\ 0 \\ c \end{pmatrix}}_{\in E_1} + \underbrace{\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}}_{\in E_2}.$$

Conclusion:

$$\mathbb{R}^3 = E_1 \oplus E_2.$$

Example 1.9

The vector subspaces of $\mathbb{R}^3$:

$$F = \{(a, a, a) \in \mathbb{R}^3 \mid a \in \mathbb{R}\} \quad \text{and} \quad G = \{(x, y, z) \in \mathbb{R}^3 \mid x = y - 2z\}$$

are in direct sum, since if $(a, a, a) \in G$, then

$$a = a - 2a \quad \Rightarrow \quad a = 0,$$

and therefore

$$F \cap G = \{(0,0,0)\}.$$

Proposition 1.4: 1.12

Let E be a vector space over the field $\mathbb{K}$. Every vector subspace of E has at least one complementary vector subspace.

1.3 Generated subspace

Theorem 1.1

Let $v_1, \ldots, v_n$ be a finite set of vectors in a vector space E over a field $\mathbb{K}$. Then:

1. The set of all linear combinations of $v_1, \ldots, v_n$ is a subspace of E.
2. It is the smallest subspace of E (with respect to inclusion) containing $v_1, \ldots, v_n$.

Proof 1.2

1. Let

$$W = \{\lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_i \in \mathbb{K}\}.$$

   To show that W is a subspace:

   - **Closed under addition:** If $u, w \in W$, then $u + w$ is also a linear combination of $v_1, \ldots, v_n$.
   - **Closed under scalar multiplication:** If $\alpha \in \mathbb{K}$ and $w \in W$, then $\alpha w \in W$.
   - **Contains zero:** $0 = 0v_1 + \cdots + 0v_n \in W$.

2. Let U be any subspace of E containing $v_1, \ldots, v_n$. Since U is closed under linear combinations, we have $W \subseteq U$. Thus, W is the smallest such subspace.

Notation 1.1

This subspace is called the subspace generated by $v_1, \dots, v_n$, and is denoted $\mathrm{Span}(v_1, \dots, v_n)$. Therefore, we have:

$$u \in \mathrm{Span}(v_1, \dots, v_n) \iff \exists \lambda_1, \dots, \lambda_n \in \mathbb{K} \text{ such that } u = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

Remark 1.4

1. Saying that $\mathrm{Span}(v_1, \dots, v_n)$ is the smallest subspace of E containing the vectors $v_1, \dots, v_n$ means that if F is a subspace of E containing $v_1, \dots, v_n$, then
$$\mathrm{Span}(v_1, \dots, v_n) \subseteq F.$$
2. More generally, we can define the subspace spanned by any subset V (not necessarily finite) of a vector space: $\mathrm{Span}(V)$ is the smallest subspace containing V.

Example 1.10

1. The vector subspace of $\mathbb{R}^3$ generated by the set
$$A = \{(-2, 0, 1), (3, 1, 1)\}$$
is
$$\begin{aligned} \mathrm{Span}(A) &= \{a(-2, 0, 1) + b(3, 1, 1) \mid a, b \in \mathbb{R}\} \\ &= \{(-2a + 3b,\ b,\ a + b) \mid a, b \in \mathbb{R}\}. \end{aligned}$$
2. In $\mathbb{C}_2[X]$, let the vectors $u = iX - X^2$ and $v = 1 + i + X$. Then
$$\begin{aligned} \mathrm{Span}(u, v) &= \{\beta(iX - X^2) + \gamma(1 + i + X) \mid \beta, \gamma \in \mathbb{C}\} \\ &= \{\gamma(1 + i) + (\beta i + \gamma)X - \beta X^2 \mid \beta, \gamma \in \mathbb{C}\}. \end{aligned}$$

Theorem 1.2

1. If a vector w is a linear combination of the vectors $(v_1, \dots, v_n)$, then
$$\mathrm{Span}(v_1, \dots, v_n, w) = \mathrm{Span}(v_1, \dots, v_n).$$

2. If a vector w is a linear combination of the vectors $(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$, then

$$\text{Span}(v_1, \dots, v_{k-1}, v_k + w, v_{k+1}, \dots, v_n) = \text{Span}(v_1, \dots, v_n).$$

Proof 1.3

1. For part (1):

   - ($\subseteq$) Any vector in $\text{Span}(v_1, \dots, v_n, w)$ can be written as

   $$\sum_{i=1}^{n} \lambda_i v_i + \mu w.$$

   Since w is a linear combination of $v_1, \dots, v_n$, this reduces to a linear combination of $v_1, \dots, v_n$.

   - ($\supseteq$) Trivial, since $\{v_1, \dots, v_n\} \subseteq \{v_1, \dots, v_n, w\}$.

2. For part (2): Let

$$S_1 = \text{Span}(v_1, \dots, v_{k-1}, v_k + w, v_{k+1}, \dots, v_n), \quad S_2 = \text{Span}(v_1, \dots, v_n).$$

   - ($S_1 \subseteq S_2$): Since

   $$v_k + w = v_k + \sum_{i \neq k} \alpha_i v_i,$$

   it follows that $v_k + w \in S_2$. Hence $S_1 \subseteq S_2$.

   - ($S_2 \subseteq S_1$): We can write

   $$v_k = (v_k + w) - w,$$

   where $w \in S_1$ by construction. Therefore, $v_k \in S_1$, and all other $v_i \in S_1$. Thus $S_2 \subseteq S_1$.

Theorem 1.3

Let A and B be two subsets of a $\mathbb{K}$-vector space E. Then:

1. $A \subseteq B \implies \text{Span}(A) \subseteq \text{Span}(B)$.

2. $\text{Span}(A \cup B) = \text{Span}(A) + \text{Span}(B)$.

Proof 1.4

1. If $A \subseteq B$, then any linear combination of elements from A is also a linear combination of elements from B. Thus $\text{Span}(A) \subseteq \text{Span}(B)$.

2. For the equality:

   - ($\subseteq$) Any vector in $\text{Span}(A \cup B)$ can be written as a linear combination of vectors from both A and B, hence it belongs to $\text{Span}(A) + \text{Span}(B)$.
   - ($\supseteq$) Since $A, B \subseteq A \cup B$, by part (1) we have $\text{Span}(A), \text{Span}(B) \subseteq \text{Span}(A \cup B)$, and thus their sum is also contained in $\text{Span}(A \cup B)$.

Definition 1.6

A family of vectors $S = \{v_i,\ i \in I\}$ in a vector space E is called a **generating family** (or **spanning set**) if every vector in E can be expressed as a linear combination of the vectors in S, i.e.,

$$\text{Span}(S) = E.$$

Mathematically, S generates E if

$$\forall v \in E,\ \exists\, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}, \quad \text{such that} \quad v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

Example 1.11

In $\mathbb{R}^3$, the set

$$\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

is a generating family of $\mathbb{R}^3$.
Also, the set

$$\{(1,1,0),\ (0,1,1),\ (1,0,1)\}$$

is a generating family as well.

Example 1.12

Consider the vector space of polynomials of degree at most 4, denoted by $\mathbb{R}_4[X]$, which consists of all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

A generating set for this space is the set

$$\{1, x, x^2, x^3, x^4\}.$$

Remark 1.5

The generating set of a $\mathbb{K}$-vector space is not unique.

Definition 1.7: Free Family (Linearly Independent)

Let E be a $\mathbb{K}$-vector space and let $\{v_1, \ldots, v_n\}$ be a family of vectors in E. We say that the family $\{v_1, \ldots, v_n\}$ is **free** (or **linearly independent**) in E if

$$\forall \lambda_1, \ldots, \lambda_n \in \mathbb{K}, \quad \lambda_1 v_1 + \cdots + \lambda_n v_n = 0_E \quad \Rightarrow \quad \lambda_1 = \cdots = \lambda_n = 0_{\mathbb{K}}.$$

Example 1.13: Standard basis in $\mathbb{R}^n$

The canonical basis vectors form a free family in $\mathbb{R}^n$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

This family is free because the equation

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = 0$$

implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Example 1.14: Polynomials in $\mathbb{R}[x]$

The family $\{1, x, x^2, \dots, x^n\}$ is free in $\mathbb{R}[x]$.
Indeed, for any linear combination

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0,$$

we must have

$$a_0 = a_1 = \cdots = a_n = 0,$$

since a nonzero polynomial of degree at most n cannot vanish identically.

Definition 1.8: Dependent Family

Let E be a $\mathbb{K}$-vector space and let $\{v_1, \dots, v_n\}$ be a family of vectors in E.
We say that the family $\{v_1, \dots, v_n\}$ is **dependent** (or **linearly dependent**) if it is not free. This means that there exist scalars $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, not all zero, such that

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0_E.$$

Example 1.15: Linearly dependent vectors in $\mathbb{R}^2$

Consider the following two vectors in $\mathbb{R}^2$:

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

These vectors are linearly dependent because

$$v_2 - 2v_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Theorem 1.4: Characterization of Linear Dependence

Let $n \geq 2$. The family $\{v_1, v_2, \dots, v_n\}$ is linearly dependent if and only if one of the vectors $v_1, v_2, \dots, v_n$ is a linear combination of the others.

1.4 Basis

Definition 1.9: Basis

A family of vectors $\{x_i\}_{i\in I}$ in a vector space E over a field $\mathbb{K}$ is called a **basis** of E if the following two conditions are satisfied:

1. The family is linearly independent.
2. The family spans E; i.e., every vector in E can be expressed as a linear combination of the vectors x_i.

Example 1.16: Examples of bases

1. The set $\{(1,0),(0,1)\}$ is the standard basis for $\mathbb{R}^2$.
2. The polynomials $\{1,t,t^2\}$ form a basis for the space of quadratic polynomials $\mathbb{R}_2[X]$.

Remark 1.6

Observations:

1. The empty set $\varnothing$ is a basis for the zero vector space $\{0\}$.
2. The basis of a given vector space is not unique.

Theorem 1.5: Unique Representation

Let $B=\{e_1,e_2,\ldots,e_n\}$ be a basis for a vector space E over a field $\mathbb{K}$. Then every vector $v\in E$ can be written uniquely as

$$v=\sum_{i=1}^{n}\lambda_i e_i, \quad \lambda_i\in\mathbb{K}.$$

The scalars λ_i are called the **coordinates** of v with respect to the basis B.

Proposition 1.5: Properties of Linear Independence and Span

1. A singleton $\{x\}$ is linearly independent if and only if $x \neq 0$.
2. Any family that contains a spanning set is itself a spanning set.
3. A subset of a linearly independent set is linearly independent.
4. Any set containing a linearly dependent subset is itself linearly dependent.
5. If a set $\{v_1, \ldots, v_p\}$ contains the zero vector, it is linearly dependent.

Theorem 1.6: Basis of a Direct Sum

Let E be a vector space over $\mathbb{K}$, and let F and G be nontrivial subspaces of E (i.e., $F, G \neq \{0\}$).

1. If B_F is a basis for F and B_G is a basis for G, then $B_F \cup B_G$ spans $F + G$.
2. If $F \cap G = \{0\}$ (direct sum), then $B_F \cup B_G$ is a basis for $F \oplus G$.

Example 1.17: Direct Sum in $\mathbb{R}^4$

Consider the following vector subspaces in $\mathbb{R}^4$:

$$F = \text{Span}\{(1, -1, 0, 2)\}, \quad G = \text{Span}\{(-2, 5, 3, 1), (1, 1, -2, -2)\}.$$

Since $(1, -1, 0, 2) \neq 0$ in $\mathbb{R}^4$, and the vectors $(-2, 5, 3, 1)$ and $(1, 1, -2, -2)$ are linearly independent, it follows that:

- $\{(1, -1, 0, 2)\}$ is a basis for F,
- $\{(-2, 5, 3, 1), (1, 1, -2, -2)\}$ is a basis for G.

Therefore, the set

$$\{(1, -1, 0, 2),\ (-2, 5, 3, 1),\ (1, 1, -2, -2)\}$$

is a generating set for $F + G$.

1.5 Finite-Dimensional Vector Spaces

Definition 1.10: Finite and Infinite Dimensional Spaces

1. A vector space E over a field $\mathbb{K}$ is called **finite-dimensional** if it has a finite generating set; that is, there exists a finite family of vectors $\{v_1, v_2, \ldots, v_n\}$ in E such that $\text{Span}\{v_1, v_2, \ldots, v_n\} = E$.

2. A vector space E is called **infinite-dimensional** if it is not finite-dimensional; this means that every generating set of E is infinite.

Example 1.18

1. The vector spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ and $\mathbb{R}_n[X]$ for $n \in \mathbb{N}$ are finite-dimensional.

2. The vector space $\mathbb{R}[X]$ is infinite-dimensional.

Theorem 1.7: Dimension of a Vector Space

Let E be a finite-dimensional vector space over a field $\mathbb{K}$. Then all bases of E have the same cardinality. This common number is called the **dimension** of E, and is denoted by $\dim(E)$ or $\dim_{\mathbb{K}}(E)$.

Theorem 1.8: Existence of a Basis

Every non-trivial finite-dimensional vector space $E \neq \{0\}$ over a field $\mathbb{K}$ admits a basis.

Corollary 1.1: Properties of Dimension

Let E be a finite-dimensional vector space over a field $\mathbb{K}$ with $\dim(E) = n$. Then:

1. Every linearly independent set in E has at most n elements.

2. Every generating set of E has at least n elements.

Remark 1.7: Basis Verification in Finite-Dimensional Spaces

Let E be a finite-dimensional vector space with $\dim(E) = n$. To show that a set of n vectors is a basis of E, it is sufficient to prove that the set is either linearly independent or a generating set for E.

Theorem 1.9: Incomplete Basis Theorem

Let E be a finite-dimensional vector space over a field $\mathbb{K}$, and let L be a linearly independent subset of E. Then there exists a basis B of E such that $L \subseteq B$ and B has finite cardinality.

Theorem 1.10: Grassmann's Formula

Let E be a finite-dimensional vector space over a field $\mathbb{K}$, and let F and G be two subspaces of E. Then

$$\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

In particular, F and G are in direct sum (i.e., $F + G = F \oplus G$) if and only if

$$\dim(F+G) = \dim(F) + \dim(G).$$

Theorem 1.11: Subspace Dimension Theorem

Let E be a finite-dimensional vector space. If F is a subspace of E, then:

1. F is finite-dimensional.
2. $\dim(F) \leq \dim(E)$.
3. $\dim(F) = \dim(E)$ if and only if $F = E$.

Remark 1.8

The dimension of the zero vector space is 0.

Example 1.19: Finite and Infinite Dimensional Spaces

1. $\dim(\mathbb{K}^n) = n$.
2. $\dim(\mathbb{K}_n[X]) = n + 1$.
3. $\dim(\mathbb{K}[X])$ is infinite.
4. $\dim_{\mathbb{C}} \mathbb{C} = 1$ and $\dim_{\mathbb{R}} \mathbb{C} = 2$.

Theorem 1.12: Characterization of Supplementary Subspaces

Let F and G be two subspaces of a finite-dimensional vector space E. Then F and G are supplementary in E (i.e., $E = F \oplus G$) if and only if any two of the following three conditions hold:

1. $\dim F + \dim G = \dim E$,
2. $F \cap G = \{0\}$,
3. $F + G = E$.