Chapter 5 : vibration with two degree of freedom

V.1 Introduction

Multiple degrees of freedom systems is defined as systems that require several independent coordinates to specify their positions. The number of degrees of freedom determines the eigen modes.

V.2 Free two-degree of freedom systems

Systems that require two independent coordinates to specify their positions are called two-degree of freedom systems, two-degree of freedom systems are made up of two coupled one-degree-of-freedom systems.

V.2.1 Types of coupling

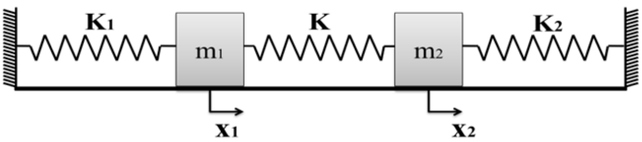
There are three types of coupling: elastic, inertial and viscous.

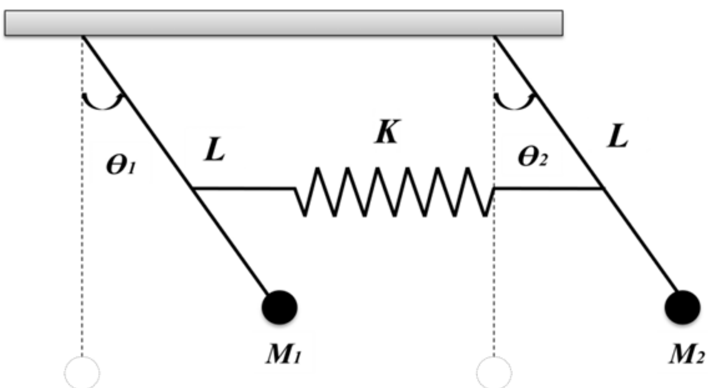
V.2.1.1 Elastic Coupling:

Coupling in mechanical systems is provided by elasticity (a spring). In electrical systems, we find circuits coupled by capacitance, which is equivalent to elastic coupling

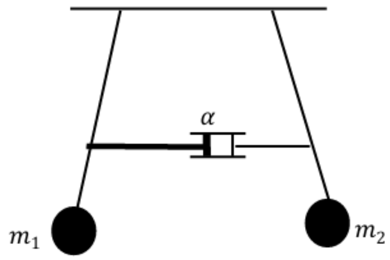
V-2. Example of free coupled oscillators

V-2.1. In mechanics.





3) We can also have the coupling by viscosity: the connection is due to a viscous mechanical friction:



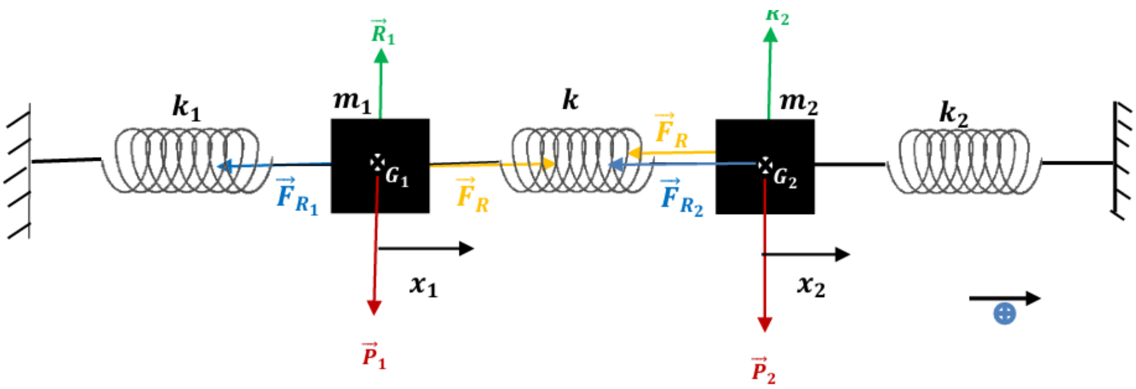
V-3. System with 2 degrees of freedom

To study systems with two degrees of freedom, it is necessary to write two differential equations of motion that can be obtained from the LaGrange equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = 0 \end{cases}$$

A 2-degree system has 02 generalized coordinates 02 differential equations and 02 proper pulsations ω ω .

<u>**MASS SPRING**</u>



When this system is moved away from its equilibrium position and then left to itself, it performs a free vibratory movement.

$$E_c = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$E_P = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K (x_2 - x_1)^2 + \frac{1}{2} K_2 (-x_2)^2$$

L=Ec−EP

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} K_1 x_1^2 - \frac{1}{2} K (x_2 - x_1)^2 - \frac{1}{2} K_2 x_2^2$$

The Lagrange equations in this case are written:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \end{cases}$$

$$\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\left(\frac{\partial L}{\partial x_1}\right) = -(K_1 + K) x_1 + K x_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\left(\frac{\partial L}{\partial x_2}\right) = -(K_2 + K) x_2 + K x_1$$

The equations describing the variation of the elongations x1 and x2 as a function of time are written as follows:

$$\begin{cases} m_1\ddot{x}_1 + (K_1 + K)x_1 - Kx_2 = 0 \\ m_2\ddot{x}_2 + (K_2 + K)x_2 - Kx_1 = 0 \end{cases}$$

The terms: −Kx1 and −Kx2 are called: Coupling Terms

We assume that the system admits harmonic solutions:

$$\text{Donc :} \begin{cases} x_1(t) = A_1 \cos(\omega t + \varphi_1) \Rightarrow \ddot{x}_1 = -\omega^2 x_1 \\ x_2(t) = A_2 \cos(\omega t + \varphi_2) \Rightarrow \ddot{x}_2 = -\omega^2 x_2 \end{cases}$$

By replacing the solutions in the differential system. We obtain a linear system following :

$$\begin{cases} -\omega^2 m_1\ddot{x}_1 + (K_1 + K)x_1 - Kx_2 = 0 \\ -\omega^2 m_2\ddot{x}_2 + (K_2 + K)x_2 - Kx_1 = 0 \end{cases} \Rightarrow \begin{cases} (-m_1\omega^2 + K_1 + K)x_1 - Kx_2 = 0 \\ -Kx_1 + (-m_2\omega^2 + K_2 + K)x_2 = 0 \end{cases} \ldots\ldots(\mathbf{1})$$

$$\begin{pmatrix} -m_1\omega^2 + K_1 + K & -K \\ -K & -m_2\omega^2 + K_2 + K \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system admits non-zero solutions if only if the determinant = 0

$$\Delta(\omega) = \begin{vmatrix} -m_1\omega^2 + K_1 + K & -K \\ -K & -m_2\omega^2 + K_2 + K \end{vmatrix} = 0$$

The determinant Δ(ω) is called the characteristic determinant. The equation Δ(ω) = 0 is called the characteristic equation or the equation with proper pulsations. It is written

$$\Delta(\omega) = (K_1 + K - m_1\omega^2) \times (K_2 + K - m_2\omega^2) - K^2 = 0$$

$$\omega^4 - (\Omega_1^2 + \Omega_2^2)\omega^2 + \Omega_1^2\Omega_2^2(1 - K'^2) = 0$$

The following constants are defined as follows:

$$\Omega_1^2 = \frac{K_1}{m_1}, \ \Omega_2^2 = \frac{K_2}{m_2}, \ K'^2 = \frac{K^2}{(K_1+K)(K_2+K)}$$

K' is called the coupling coefficient

The two natural pulsations are:

$$\begin{cases} \omega_1^2 = \dfrac{\Omega_1^2 + \Omega_2^2}{2} - \dfrac{1}{2}\sqrt{(\Omega_1^2 - \Omega_2^2)^2 + 4K\Omega_1^2\Omega_2^2} \\ \omega_2^2 = \dfrac{\Omega_1^2 + \Omega_2^2}{2} + \dfrac{1}{2}\sqrt{(\Omega_1^2 - \Omega_2^2)^2 + 4K\Omega_1^2\Omega_2^2} \end{cases}$$

A1,A2,B1,B2,ϕ1 and ϕ2 are integration constants determined from the initial conditions. In order to simplify the number of unknowns; We determine the ratios of amplitudes to the own modes

$$\begin{cases} \omega = \omega_1;\ \dfrac{A_1}{A_2} \\ \omega = \omega_2;\ \dfrac{B_1}{B_2} \end{cases}$$

In the case of a symmetric system m1 = m2 = m and K1 = K2 = k , relations (1)

become:

$$\begin{cases} (-m\omega^2 + k + K)x_1 - Kx_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \\ -Kx_1 + (-m\omega^2 + K + k)x_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{cases}$$

$$\begin{pmatrix} -m\omega^2 + k + K & -K \\ -K & -m\omega^2 + K + k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system admits non-zero solutions if only if the determinant = 0

$$\Delta(\omega) = \begin{vmatrix} -m_1\omega^2 + K_1 + K & -K \\ -K & -m_2\omega^2 + K_2 + K \end{vmatrix} = 0$$

$$\Delta(\omega) = (-m\omega^2 + k + K)^2 - K^2 = 0$$

$$\begin{cases} -m\omega^2 + k + K = K \\ -m\omega^2 + k + K = -K \end{cases} \Rightarrow \omega_1^2 = \frac{k}{m},\ \omega_2^2 = \frac{k+2K}{m}$$

The two proper pulsations are

$\omega_1 = \sqrt{\frac{k}{m}}$ et $\omega_2 = \sqrt{\frac{k+2K}{m}}$

The solutions of the system:

The general solution is then written as a linear combination of the two solutions.

$$\begin{cases} x_1(t) = A_1 \sin(\omega_1 t + \varphi_1) + B_1 \sin(\omega_2 t + \varphi_2) \\ x_2(t) = A_2 \sin(\omega_1 t + \varphi_1) + B_2 \sin(\omega_2 t + \varphi_2) \end{cases}$$