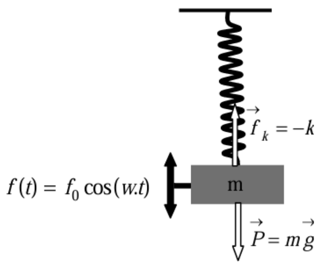
IV.1 Introduction

Forced vibrations (oscillations) occur when the system is subjected along its vibrations to one of the periodic external forces. Often, these forces are called external excitations. The resulting motion is called the response of the system to the external excitation. The excitation force can be harmonic, periodic non-harmonic, non-periodic or random. In this course, we are only interested in harmonic excitations. The harmonic excitation can be given mathematically by:

$$f(t) = f_0 \sin(w.t+\varphi), \quad f(t) = f_0 \cos(w.t+\varphi), \quad f(t) = f_0 e^{i(w.t+\varphi)}.$$

Undamped system:

Often, mechanical systems are not undergoing free vibration, but are subject to some applied force that causes the system to vibrate. In this section, we will consider only harmonic (that is, sine and cosine) forces, but any changing force can produce vibration. When we consider the free-body diagram of the system, we now have an additional force to add, namely the external harmonic excitation.

The equation of motion of the system will be: $\ddot{x} + \omega_0^2 x = f_0 \cos(w_e t)$ his equation of motion for the system can be re-written in standard form: $\ddot{x} + \omega_0^2 x = \dfrac{f_0}{m} \cos(w_e t)$	

The solution to this system consists of the superposition of two solutions: a particular solution, x_p (related to the forcing function), and a complementary solution, x_h (which is the solution to the system without forcing).

$$x(t) = x_h(t) + xp(t)$$

We can obtain the particular solution by assuming a solution of the form:

$$x_P(t) = B\, cos(w_e t + \varphi_e)$$

Where ω_e is the frequency of the harmonic forcing function. We differentiate this form of the solution, and then sub into the above equation of motion:

$$\dot{x} = -Aw_e\, sin(we + \varphi e)$$

$$B\cos(W_e t+\varphi e)(W_0^2-W_e^2)=\frac{f_0}{m}\cos(W_e t)$$

By comparison:

$\frac{f_0}{m}=B(w_0^2-W_e^2)\Rightarrow B=\frac{f_0}{m(w_0^2-w_e^2)}$ and $\varphi e=0$

Dividing by k:

$$\Rightarrow B=\frac{f_0/_k}{\frac{m}{k}(w_0^2-w_e^2)}=\frac{f_0/_k}{1-\frac{w_e^2}{w_0^2}}$$

Thus, the general solution for a forced, undamped system is:

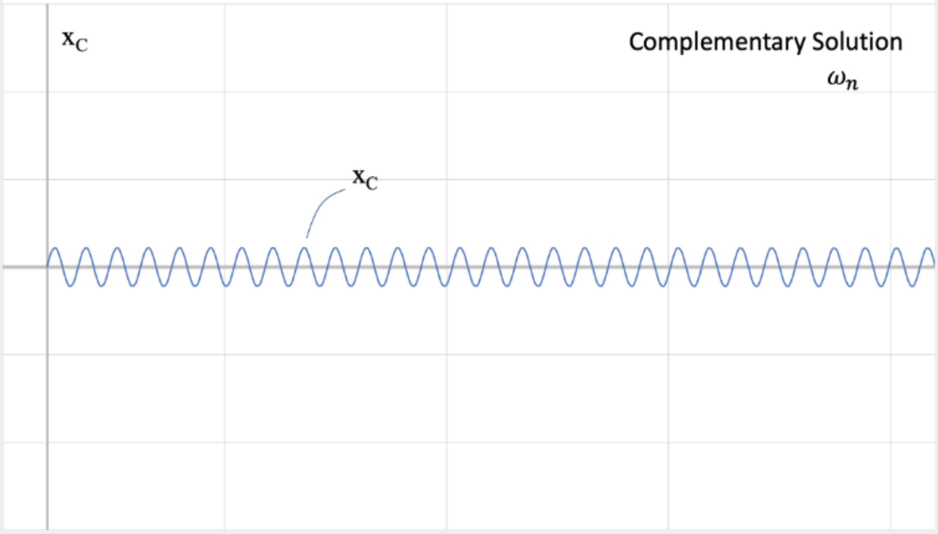
$$x(t)=A\,cos(w_0 t+\varphi)+\frac{f_0/_k}{1-\frac{w_e^2}{w_0^2}}\cos(w_e t)$$

A and φ determined from the initial condition at (t=0)

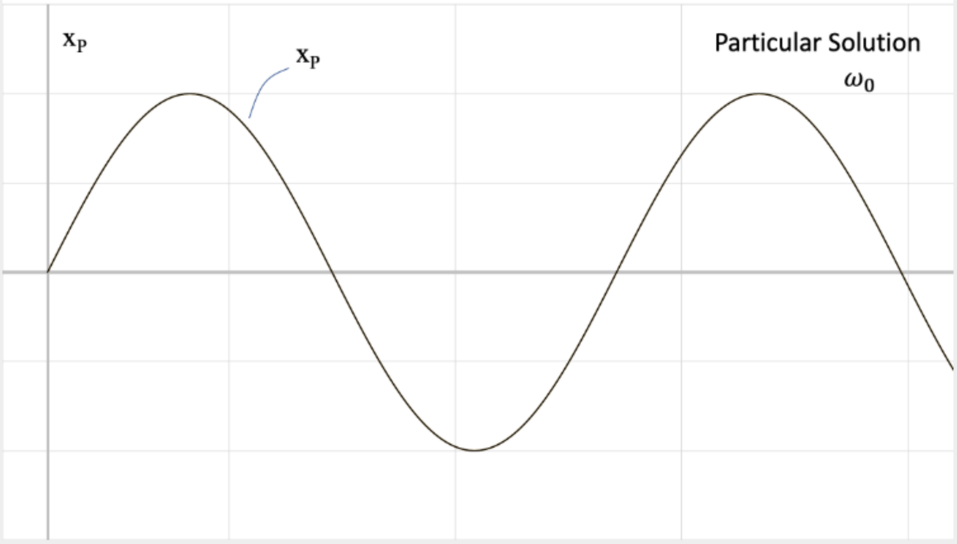
$$At\ t=0\ \begin{cases}x(0)=0\\ \dot{x}(0)=0\end{cases}$$

After calculations we find that $\begin{cases}\varphi=0^0\\ A=-\frac{f_0/_k}{1-\frac{w_e^2}{w_0^2}}\end{cases}$

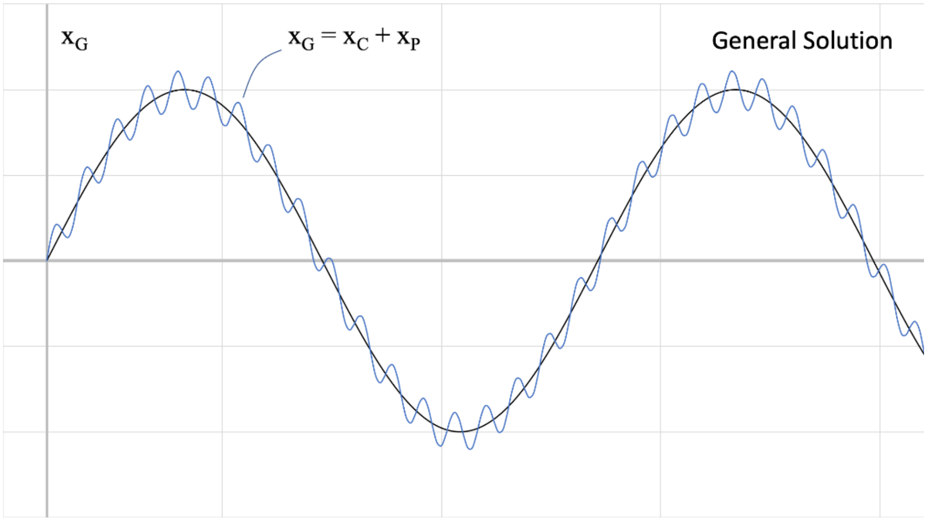
So finally: x(t) = $\frac{f_0/_k}{1-\frac{w_e^2}{w_0^2}}(\cos w_e t-cosw_0 t)$



The complementary solution of the equation of motion. This represents the natural response of the system, and oscillates at the angular natural frequency. This is the transient response.



The particular solution of the equation of motion. This represents the forced response of the system, and oscillates at the angular forced frequency. This is the steady-state response.



The above figures show the two responses at different frequencies. Recall that the value of ω_0 comes from the physical characteristics of the system (m, k) and ω_e comes from the force being applied to the system. These responses are summed, to achieve the blue response (general solution) in the third figure.

Special cases:

1. **If ($w_e \rightarrow 0$) t**he excitation force became constant and doesn't vary with time so the equation of motion became:

$$x(t) = {}^{f_0}/_{k}\,(1 - \cos w_0 t)$$

We know that f_0 is a static force (constant with time) so :

$\delta_{stat} = f_0 \,/\, k$ called static amplitude

So

$$\Rightarrow \frac{B(w_e)}{\delta_{stat}} = \frac{1}{1 - \frac{w_e^2}{w_0^2}}$$

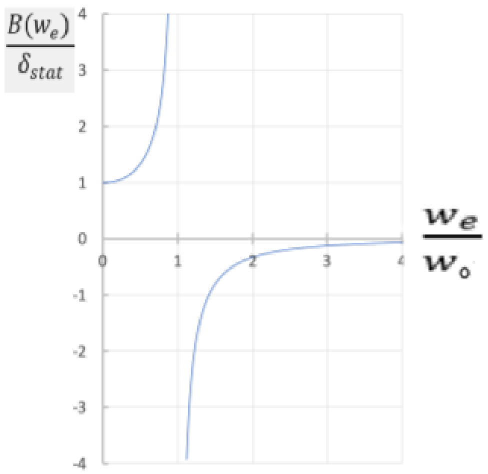
$B(w_e)$ is the dynamic amplitude and $\frac{Bw_e}{(\delta)_{stat}}$ is called the amplitude ratio

Amplification factor:

The $ratio\ x = \frac{w_e}{w_0}$ (the ratio of the dynamic amplitude to the static amplitude)

represents the amplification factor.

$$\Rightarrow \frac{B(w_e)}{\delta_{stat}} = \frac{1}{1 - \frac{w_e^2}{w_0^2}}$$

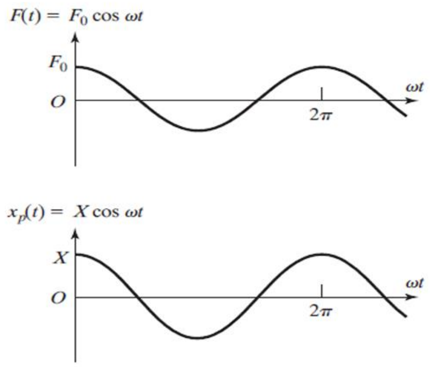


From the figure above, we can discuss various cases:

- ***ω**$_e$* < ***ω**$_0$*: Amplification(magnification) is positive and greater than 1, meaning the vibrations are in phase (when the force acts to the left, the system displaces to the left) and the amplitude of vibration is larger than the static deflection.

$$B(w_e) = \frac{\delta_{stat}}{1 - \frac{w_e^2}{w_0^2}} > 0$$

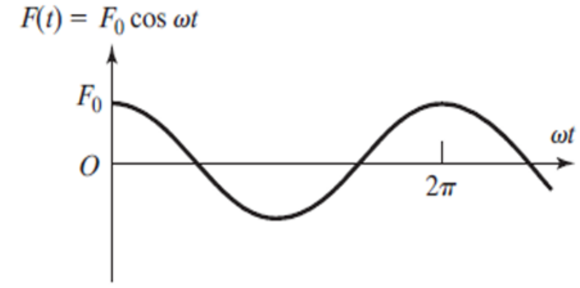
$\begin{cases} f = f_0\, cos(w_e t) \\ x_p = B\, cos(w_e t) \end{cases}$ →The system's response to excitation and excitation are said to be in phase

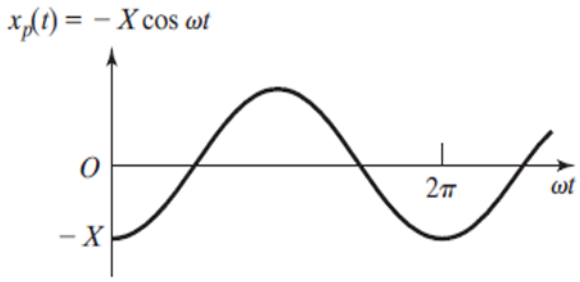


- ***ω**$_e$* > ***ω**$_0$*: Amplification(Magnification) is negative and the absolute value is typically smaller than 1, meaning the vibration is out of phase with the motion of the forcing function (when the force acts to the left, the system displaces to the right) and the amplitude of vibration is smaller than the static deflection.

$B(w_e) = \frac{\delta_{stat}}{1-\frac{w_e^2}{w_0^2}} < 0 \rightarrow -B(w_e) = \frac{\delta_{stat}}{\frac{w_e^2}{w_0^2}-1} < 0$

So $\begin{cases} f = f_0\, cos(w_e t) \\ x_p = -B\, cos(w_e t) \end{cases}$





- ***$\omega_e = \omega_o$*: resonance** occurs. When the frequency of the external excitation force is equal to the natural frequency of the system, the amplitude of the system's response increases to infinity. This phenomenon is called resonance. The study of the phenomenon of resonance is of great importance in industrial construction and engineering.

$$\frac{w_e}{w_0} = 1 \Rightarrow w_e = w_0 \;\Rightarrow\quad B(w_e) = \frac{\delta_{stat}}{1-\frac{w_e^2}{w_0^2}} \rightarrow \infty$$

We must solve the equation of motion in the case where we = w_0 ;

$$\ddot{x} + w_0^2 x = f_0\, cos(w_0 t)$$

As long as n is a solution of the characteristic equation (see TD 1), the particular solution of the non-homogeneous equation is of the form: $x_P = t(A\, cos(w_0 t + \varphi)$

To find the constants A and φ, we replace xp in the differential equation,

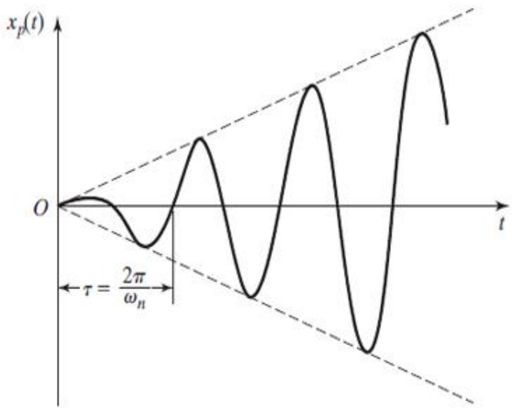
Atfer the cacule we find that :

$$\begin{cases} A = \frac{f_0 \setminus m}{2w_0} \\ \varphi = 0 \end{cases} \rightarrow A = B = \frac{\delta_{stat}}{2}$$

So the particular solution can written as:

$$x_P = t\frac{\delta_{stat}}{2}cos(w_0 t)$$

The response of the system to the force of the excitation is periodic with increasing amplitude with time.



Damped forces system:

IV.2 Differential equation of motion

The differential equation of forced oscillations of systems with one degree of freedom is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F_{ext}$$

a) For a translational movement the equation is written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F_{ext}$$

b) For a rotational motion, the equation is written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = \mathcal{M}(F_{ext})$$

- $Fext$: the force generalized to an external force.
- $\mathcal{M}(Fext)$: Moment of the applied force.

$$\boldsymbol{\mathcal{M}(Fext) = Fext \times L}$$

Moment: characterizes the ability of a force to turn an object around a point.

L: is the straight distance of action of the force.

The differential equation of forced vibratory motion is:

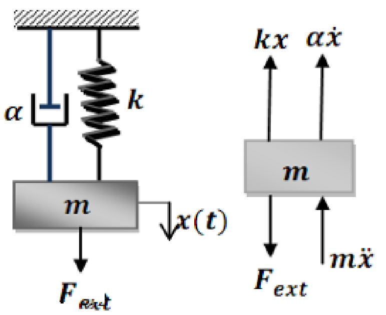
$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = A(t)$$

IV.2.1 Example of a forced damped system (mass-spring-damper system)

In the figure opposite, the mass m is fixed to a spring K and a shock absorber α.

Let's take the case of the elastic pendulum (vertical for example).
The study of the damped oscillator is done in the same way as previously but by adding an external force (**Fext**).
In one dimension, Lagrange's equation is written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = -\frac{\partial D}{\partial \dot{q}} + F_{ext}$$



Let's take a sinusoidal force applied to the mass m: **Fext**=$\boldsymbol{F_0 \cos(\omega t)}$

The kinetic energy of the system: it is the kinetic energy of the mass **m**:

$$T = \frac{1}{2} m\dot{x}^2$$

The potential energy of the system: this is the energy stored in the spring:

$$\boldsymbol{U = \frac{1}{2}k(x + x_0)^2 = \frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 + kxx_0 = \frac{1}{2}kx^2 + \frac{1}{2}kx_0^2}$$

(At equilibrium: $\frac{\partial U}{\partial x} = 0 \Rightarrow x_0 = 0$)

Dissipation function: $\boldsymbol{D = \frac{1}{2}\alpha\dot{x}^2}$

The Lagrange function: $L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m x \Rightarrow \frac{d}{dt}\left(\frac{\dot{\partial} L}{\partial \dot{x}}\right) = m\ddot{x} \\ \frac{\partial L}{\partial x} = -kx \\ \frac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \end{cases}$$

Substituting into Lagrange's equation we will have:

$m\ddot{x} - (-kx) + \alpha\dot{x} = f_0 \cos(w_e t)$ Dividing by m we find:

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos(w_e t)$$

Often the differential equation is written in the reduced form:

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = \frac{f_0}{m}\cos(w_e t)$$

Where $\begin{cases} \delta = \frac{\alpha}{2m} & damping\ factor \\ w_0^2 = \frac{k}{m} & proper\ pulsation \end{cases}$

We therefore obtain a second-order linear differential equation with constant coefficients and second member.

III.2 Solution of the differential equation of motion

The general solution of this differential equation is the sum of two terms:

- A solution of the equation without a second member: homogeneous solution $x_h(t)$
- A solution of the equation with a second member: particular solution $x_p(t)$

The total solution of the equation of motion will therefore be: $\boldsymbol{x(t)}=\boldsymbol{x_h(t)}+\boldsymbol{x_p(t)}$

III.2.1 Homogeneous solution: The homogeneous solution corresponds to the solution of the differential equation without a second member

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$$

It appears that the solution of the homogeneous differential equation is simply the solution found for the damped harmonic oscillator in a free regime in the case of weakly damped oscillations.

$$x_h(t) = A_e^{-\delta t} \, cos(wt + \varphi) \; with \; \omega = \sqrt{w_0^2 - \delta^2}$$

Note: The general solution of the equation without second member corresponds to a transient regime (which only lasts a certain time).

$$x(t) = \begin{cases} C_1 e^{-\delta t + \sqrt{\delta^2 - \omega_0^2}} + C_2 e^{-\delta t - \sqrt{\delta^2 - \omega_0^2}} & (si\ \delta > \omega_0) \\ (C_1 + C_2 t) e^{-\delta t} & (si\ \delta = \omega_0) \\ A e^{-\delta t} \sin(\omega_a t + \phi) \, , \; \omega_a = \sqrt{\omega_0^2 - \delta^2} & (si\ \delta < \omega_0) \end{cases} \quad + \quad \boldsymbol{X \cos(\omega t + \Phi)}$$

III.2.2 Particular solution:

When the component $\boldsymbol{x}$h($\boldsymbol{t}$) becomes truly negligible, there remains only the particular solution, which is the solution imposed by the excitation function. We say that we are in a forced regime or permanent regime.

The excitatory force forces the mechanical system to follow a temporal evolution equivalent to its own. So if **Fext** is a sinusoidal function

of pulsation $\boldsymbol{\omega}$; then the particular solution $\boldsymbol{xp(t)}$.

The oscillations of the mass are not necessarily in phase with the excitatory force and present a noted phase shift. The particular solution corresponding to the steady state is written as $\boldsymbol{x}$p($\boldsymbol{t}$) = $\boldsymbol{A} \cos(\boldsymbol{\omega_e t} + \boldsymbol{\varphi})$

For practical reasons, it is convenient to use complex notation. The

complex quantity associated with x(t) is written as:

$$\boldsymbol{x_P(t) = A\, e^{j(\omega t + \varphi)} \quad \text{Et} \quad F_{ext} = F_0 e^{j\omega t}}$$

<u>**Calculation of the amplitude A**</u>

The equation of motion becomes:

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = \frac{f_0}{m}\cos(w_e t) = Be^{jw_e t}$$

- We calculate the first derivative and then the second derivative:

$x_P(t) = A\,e^{j(\omega t+\varphi)} \rightarrow \dot{x}_P(t) = Aj\omega_e\,e^{j(\omega t+\varphi)} = j\omega_e x_P(t)$

$\ddot{x}p(t) = Aj^2\omega_e^2\,e^{j(\omega t+\varphi)} = -\omega^2 x_P(t)$

By replacing in above equation:

$-\omega_e^2 x_P(t) + 2\delta j\omega_e xP(t) + \omega_0{}^2 x_P(t) = Be^{jw_et} \rightarrow [(\omega_0{}^2 - \omega_e^2)+2\delta\omega j]x_P(t) =$

$[(\omega_0^2 - \omega_e^2)+2\delta\omega j]A\,e^{j(\omega t+\varphi)} = Be^{jw_et} \rightarrow [(\omega_0^2 - \omega_e^2)+2\delta\omega j]Ae^{j\varphi} = B$

We divide on "$e^{j\varphi}$" and we find:

$[(\omega_0^2 - \omega_e^2)+2\delta\omega j]A = Be^{-j\varphi}$............(1)

The conjugate of this equation is as follows:

$[(\omega_0^2 - \omega_e^2)-2\delta\omega j]A = Be^{j\varphi}$...............(2)

$(1)\times(2) \rightarrow A2[(\omega_0^2 - \omega_e^2)^2 + (2\delta\omega_e)2] = B^2 \rightarrow A = \frac{B}{\sqrt{(w_0^2 - w_e^2)^2 + (2\delta\omega e)^2}}$

III.2.2.2 Calculation of φ:

$[(\omega_0{}^2 - \omega_e^2) + 2\delta\omega_e j]A = \begin{cases} e^{-j\varphi} \\ \mathrm{B}(cos\varphi - j\sin\varphi) \end{cases} \Rightarrow tg\varphi\left(\frac{-2\delta w_e}{w_0^2 - w_e^2}\right)$

$\varphi = Arctg\left(\frac{-2\delta w_e}{w_0^2 - w_e^2}\right)$

finally : $x_P(t) = \frac{B}{\sqrt{(w_0^2 - w_e^2)^2 + (2\delta\omega e)^2}} \cos\left(w_e t + \frac{-2\delta w_e}{w_0^2 - w_e^2}\right)$

III.3 .3 Resonance Phenomenon and Quality Factor

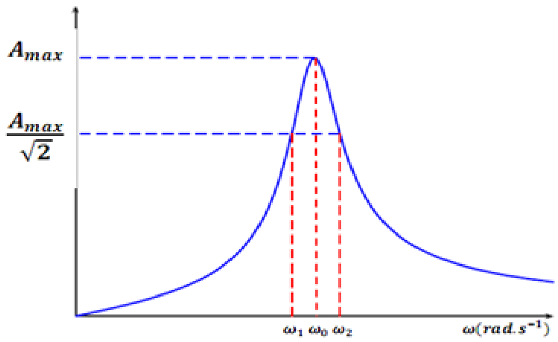
In electrical systems, this phenomenon makes it possible to calculate the quality factor Q which increases when the maximum amplitude increases:

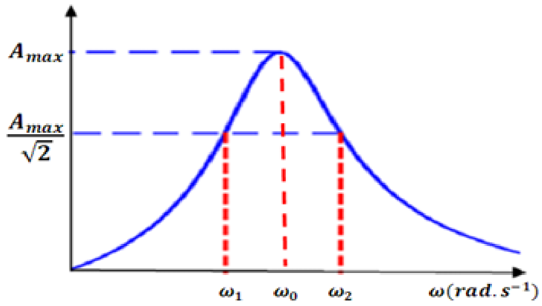
$$Q = \frac{A_{max}}{A_0} \approx \frac{1}{2\xi};\ (\omega \approx \omega_0 \approx \omega_r)$$

Another practical method to determine the quality factor

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

To characterize the sharpness (intensity) of the response of an oscillator as a function of the pulsation, we define a bandwidth: ***ω2 – ω1***





Conclusions:

When ξ *increases* $\Longrightarrow$ Q *decreases* $\Longrightarrow$ $\omega 2-\omega 1$ *increases* $\Longrightarrow$ *the resonance curve is wider* $\Longrightarrow$ *decrease in the resonance amplitude and therefore in the quality too.*

The ends of the bandwidth correspond to a speed amplitude $\sqrt{2}$

times smaller than at resonance.