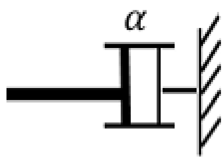
Chapter III: Free-damped vibration with one degree of freedom

III-1. Damped free oscillation

A part of the oscillator's energy is released to the external environment (dissipated by friction or radiation). The amplitude of the oscillations decreases over time and the oscillator eventually stops.

II.2 Friction force

A system subjected to friction forces is said to be: a damped system, the shock absorber is schematized and modeled by the following diagram:



- The simplest friction to model is viscous friction which depends on the speed of the object, it is given in the following form:

$$f = -\alpha v = -\alpha \dot{q}$$

α: The coefficient of viscous friction (kg/s)

v: body speed

Note: The minus sign (-) comes from the fact that the friction force opposes the motion by acting in the direction and the opposite sense to the speed.

III.3. Lagrange equation for damped systems

Taking into account the fluid friction type force (viscous friction coefficient), the Lagrange equation in this case becomes:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = F_q$$

By introducing the dissipation function: $D = {}^{1}/_{2}\, \alpha \dot{q}^2$

$$f = -2\alpha \dot{q} = -\frac{\partial D}{\partial \dot{q}}$$

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The Lagrange equation of a damped system is written with (q=x, y, z, …)

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \left[\frac{\partial L}{\partial q}\right] + \frac{\partial D}{\partial \dot{q}} = 0$$

The form of the differential equation is:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0$$

Where:

δ: is the damping coefficient.

ω_0: the natural pulsation.

III.4 Regimes of the damped oscillator

The differential equation of a damped oscillator is:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0.$$

the following characteristic equation:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$$

There are three possible regimes:

1. **Aperiodic regime ($\delta > \omega_0$,$\Delta>0$)**, friction is significant, the value of the damping coefficient is large, the system slowly returns to its equilibrium position without oscillating and the solution to the differential equation is of the form:

$$q(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$q(t) = A_1 e^{\left[-\delta - \sqrt{\delta^2 - \omega_0{}^2}\right]t} + A_2 e^{\left[-\delta + \sqrt{\delta^2 - \omega_0{}^2}\right]t}$$

A1 *et* $A2$ are constants of integration defined by the initial conditions.

2. **Critical regime (δ = ω_0, Δ=0)**, the return to equilibrium occurs rapidly without oscillation. It allows to fix the limit between the pseudo periodic and aperiodic regimes, the solution is of the form**:** **$q(t)=(A_1+A_2t)e^{-\delta t}$**
3. **Pseudoperiodic regime (δ < ω_0, Δ<0),** (low damping regime) we observe that the amplitude of the oscillations is however decreasing, it is weighted over time by a decreasing exponential factor depending on the friction, and the solution of the differential equation is of the form:

$$q(t) = Ae^{-\delta t}cos(\omega t + \varphi)$$

A and φ are integration constants determined from the initial conditions.

ω is the pseudo pulsation defined by

$$\omega = \sqrt{\omega_0^2 - \delta^2}$$

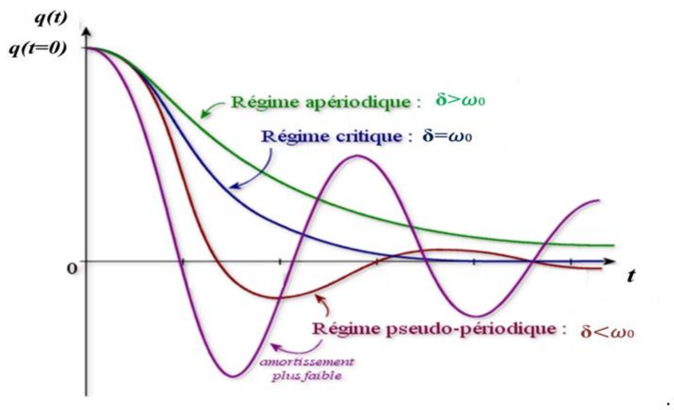
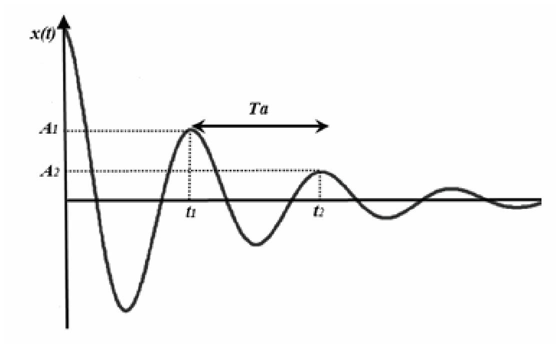


Figure III.1 illustrates the q(t) variation as a function of time t for the three regimes (aperiodic regime, critical regime and pseudo-periodic regime).

III.5 Logarithmic Decrement

It is the logarithm of the ratio of 2 successive amplitudes of the damped oscillations



$$D = ln\frac{A(t_1)}{A(t_2)} = ln\frac{A(t_1)}{A(t_1+T)} = -ln\frac{A(t_1+T)}{A(t_1)}$$

$$D = \frac{1}{n}\ln\frac{A(t)}{A(t+nT)}$$

n: the number of periods

By replacing the amplitude formulas, we obtain: $D = \delta T$

δ: is the damping coefficient.

T: is the pseudo period. It is given by: $T = \frac{2\pi}{w}$

ω is the pseudo-period and id donned by:

$w = \sqrt{w_0^2 - \delta^2}$

III.6 Quality Coefficient

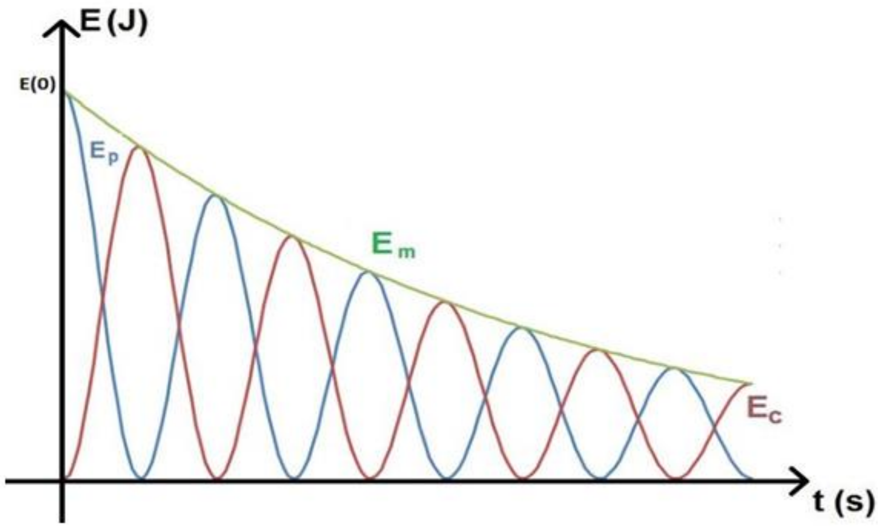
Is a unitless measure of the damping rate of an oscillator. The quality factor is used to quantify the quality of an oscillatory system. The lower the damping, the greater the quality of the system. Now Q is all the greater.

$$Q = 2\pi \cdot \frac{E_{sys}}{E_{diss}} = \frac{\omega_0}{2\,\delta}$$

III.7 Mechanical Energy: for the case of a weakly damped system ($\delta \ll \omega_0$), the mechanical energy is written:

$$E(t) = E_0 . e^{-2\delta t}$$

E0 = E (0) Energie totale initiale (à t =0), E0 = E (0) = Ec (0) + Ep (0)



The variation of total (mechanical) energy as a function of time t with friction

The damped system loses energy through dissipation phenomena (friction). The total energy of the system decreases over time.

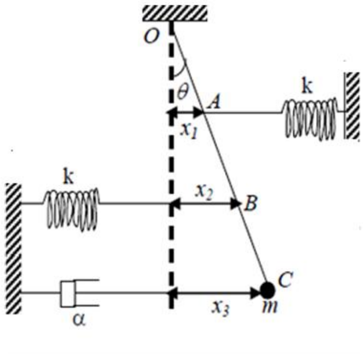
Examples of Damped Oscillators

Damping can be useful. In car suspension, damping is used to reduce the energy of the vibrations which would be caused by a car going over bumps in the road. This allows for a safer and more comfortable ride for passengers.

Speedometers on cars are damped such that when the car accelerates the needle doesn't oscillate which would lead to confusion for the driver. Sometimes, doors have a mechanism which means when they are closed, they close gently and slowly. This mechanism uses damping to slow down the door and to stop it from oscillating.

Application example:

Find the differential equation of motion and the oscillation condition in the case of small amplitudes for the system in the figure opposite with (OA = AB =BC = L/3)



The system is free damped (f(t)=0), with a single generalized coordinate (q=θ). The Lagrange equation in this case is written as follows:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = 0$$

Kinetic energy: T=$\frac{1}{2}j\dot{\theta}^2$ with j/pendulum=m(l)^2 so T=$\frac{1}{2}m(l)^2\dot{\theta}^2$

Potential energy : U=U_{k1}+U_{k2}+mgh =$\frac{1}{2}Kx_1^2 + \frac{1}{2}Kx_2^2 + mgh$ with:

$x_1 = \frac{l}{3}\sin\theta$, $x_2 = \frac{2l}{3}\sin\theta$, and $h = l - l\cos\theta = l(1-\cos\theta)$

In the case of small amplitudes the following mathematical hypotheses are considered:

$$\sin\theta \cong \theta \text{ et } \cos\theta \cong 1 - \frac{\theta^2}{2} \text{ , soit : } U = \frac{1}{2}k\left(\frac{l}{3}\theta\right)^2 + \frac{1}{2}k\left(\frac{l}{3}\theta\right)^2 + mg\frac{l}{2}\theta^2$$

$$U = \frac{1}{2}\left(\frac{5kl^2}{9} + \frac{mgl}{2}\right)\theta^2$$

Dissipation energy: $D = \frac{1}{2}\alpha(x_3)^2 = \frac{1}{2}\alpha\left(l\dot{\theta}\right)^2 = \frac{1}{2}\alpha l^2\dot{\theta}^2$ with ($l\sin\theta = l\theta$)

Lagrange:

$$L = T - U \Rightarrow \quad L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}\left(\frac{5kl^2}{9} + \frac{mgl}{2}\right)\theta^2$$

$$(\frac{\partial L}{\partial \dot{\theta}}) = ml^2 \dot{\theta} \Rightarrow \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}}) = ml^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -(\frac{5kl^2}{9} + \frac{mgl}{2})\theta$$

$$(\frac{\partial D}{\partial \dot{\theta}}) = \alpha l^2 \dot{\theta}$$

$$\Rightarrow ml^2 \ddot{\theta} + \alpha l^2 \dot{\theta} + (\frac{5kl^2}{9} + \frac{mgl}{2})\theta = 0 \Rightarrow \quad \ddot{\theta} + \frac{\alpha}{m}\dot{\theta} + (\frac{5k}{9m} + \frac{g}{l})\theta = 0$$

This is the differential equation (EDM) of the system of the form: $\ddot{q} + 2\delta\dot{q} + w_0^2 q = 0$

With $\begin{cases} \delta = \frac{\alpha}{2m} \\ w_0 = \sqrt{\frac{5k}{9m} + \frac{g}{l}} \end{cases}$

- For the system to perform an oscillatory movement it must be under-damped:

$$\lambda \prec \varpi_0 \Rightarrow \frac{\alpha}{2m} \prec \sqrt{\frac{5k}{9m} + \frac{g}{l}} \Rightarrow \alpha \prec 2m\sqrt{\frac{5k}{9m} + \frac{g}{l}}$$