Tutorial Worksheet No.3

Exercise 1.

Let $\star$ be an operation on $\mathbb{R}^*$defined by

$$\begin{aligned} \star : \mathbb{R}^* \times \mathbb{R}^* &\longrightarrow \mathbb{R}^* \\ (a,b) &\longrightarrow a \star b = \frac{1}{a} + \frac{1}{b} \end{aligned}$$

1. Calculate $2 \star 4$.
2. Is $\star$ satisfies the following properties : Identity, Inverse and Associativity.
3. Is $\star$ a binary operation ?
4. Does every elelment in $\mathbb{R}^{-\{-1,0,1\}}$ have its inverse under $\star$? If yes, then determine this element.

Exercise 2.

1. Let $u_1 = (h, 2)$, $u_2 = (1, -2h)$ and $v = (1, 2)$ are vectors in $\mathbb{R}^2$. Give value(s) for h so that v be a linear combination of $u_1$and u_2.
2. Let $u_1 = (1, 0, 1)$,and $u_2 = (0, 1, 0)$ be two elements of $\mathbb{R}^3$.
   (a) Does $\{u_1, u_2\}$span $\mathbb{R}^3$?
   (b) Are u_1 and u_2 linearly independent ?
   (c) Prove that $span\,\{u_1, u_2\}$ is a subspace of $\mathbb{R}^3$.

Exercise 3.

Let $\begin{array}{ll} u_1 = (1,1,0,0), & u_2 = (1,0,1,0) \\ u_3 = (0,0,1,1), & u_4 = (0,1,0,1) \end{array}$ and $v = (1, 2, 3, a)$, where $a \in \mathbb{R}$.

1. Determine a such that $v \in span\,\{u_1, u_2, u_3, u_4\}$
2. Is $\{u_1, u_2, u_3, u_4\}$ spans the vector space $\mathbb{R}^4$?

Exercise 4.

Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$, $H = \{(x, y, z, t) \in \mathbb{R}^4 \mid y - z = 0\}$

1. Prove that the sets W and H are subspaces of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively.
2. Determine the dimension of each subspace.

Exercise 5.

Let $C[-1,1]$ be the vector space over $\mathbb{R}$ of all continuous functions defined on the interval $[-1,1]$. Let
$W := \{f(x) \in C[-1,1] \mid f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in \mathbb{R}\}$.

1. Prove that W is a subspace of $C[-1,1]$
2. Prove that the set $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of W.

Exercise 6.

Let P_3 be the vector space of polynomials of degree 3 or less with real coefficients. Let T be a map defined by :

$$T : P_3 \longrightarrow P_3$$
$$f(x) \longrightarrow T(f(x)) = \frac{df}{dx}$$

1. Calculate $T(x)$ and $T(5x^2 + 2x^3)$
2. Prove that T is a linear transformation (linear map).

Exercise 7.

Let f be a linear map defined by :

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x, y, z) \longrightarrow f(x, y, z) = (x, x + y, x + y + z)$$

1. Determine $Ker(f)$ and $Im(f)$.
2. Determine $f \circ f$.
3. Prove that f is bijective and determine f^{-1}.

Exercise 8 (Homework).

A) Let $P[X]_{\leq n}$the set of polynomials of degree less than or equal to n with real coefficients

- Prove that $P[X]_{\leq n}$ is a vector space over $\mathbb{R}$.

B) Does the set of real matrices$M_{2\times 2}$form a vector space over $\mathbb{R}$.