Analysis I: Solutions of Tutorial Exercise Sheet 5

2025-2026

Exercise 01

Part 1: Approximation using the definition of the definite integral

The definite integral of a function $f(x)$ from $x = a$ to $x = b$ is defined as:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\big[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)\big] \tag{*}$$

where $h = \dfrac{b-a}{n}$ and $n \to \infty$.

We approximate the integral

$$I = \int_1^2 \frac{dx}{x}$$

using a finite value of n . Choose $n = 10$ (see figures in lecture notes of chapter 5), so that:

$$h = \frac{2-1}{10} = 0.1.$$

Then, according to definition (*),

$$I \approx h\big[f(1) + f(1.1) + f(1.2) + \cdots + f(1.9)\big],$$

where $f(x) = \dfrac{1}{x}$.

Compute each term:

$$\begin{aligned} f(1) &= 1.0000, & f(1.1) &= 0.9091, & f(1.2) &= 0.8333, \\ f(1.3) &= 0.7692, & f(1.4) &= 0.7143, & f(1.5) &= 0.6667, \\ f(1.6) &= 0.6250, & f(1.7) &= 0.5882, & f(1.8) &= 0.5556, \\ f(1.9) &= 0.5263. \end{aligned}$$

Sum:

$$\sum_{k=0}^{9} f(1+0.1k) = 1.0000+0.9091+0.8333+0.7692+0.7143+0.6667+0.6250+0.5882+0.5556+0.5263 = 7.1876.$$

Multiply by $h = 0.1$:

$$I \approx 0.1 \times 7.1876 = 0.7188.$$

Geometric interpretation: The sum $h\sum_{k=0}^{n-1} f(a+kh)$ represents the total area of n rectangles, each of width h, with heights given by the function values at the left endpoints of the subintervals. Since $f(x) = 1/x$ is positive and decreasing on $[1, 2]$, the top-left corner of each rectangle lies above the curve $y = 1/x$. Thus, this approximation **overestimates** the true area under the curve between $x = 1$ and $x = 2$.

Part 2: Improving the approximation

The overestimate can be balanced by also considering an underestimate. Using the right endpoints gives:

$$I \approx h\big[f(1.1) + f(1.2) + \cdots + f(2.0)\big].$$

Compute:

$$f(2.0) = 0.5000,$$

and using the previous values for $f(1.1)$ through $f(1.9)$:

$$\sum_{k=1}^{10} f(1+0.1k) = 0.9091+0.8333+0.7692+0.7143+0.6667+0.6250+0.5882+0.5556+0.5263+0.5000 = 6.6877.$$

Then:

$$I \approx 0.1 \times 6.6877 = 0.6688.$$

This right-endpoint sum **underestimates** the true area because each rectangle lies below the curve. A better estimate is obtained by taking the arithmetic mean of the two approximations:

$$I \approx \frac{0.7188 + 0.6688}{2} = 0.6938.$$

Geometrically, this average corresponds to replacing each pair of left- and right-rectangles with a trapezoid whose top side connects the points on the curve at the endpoints of each subinterval. This is known as the **trapezoidal rule**.

The exact value of the integral is $\ln 2 \approx 0.6932$. Our improved estimate 0.6938 is very close, with an error of only 0.0006. Further improvement can be achieved by increasing n (using more subintervals).

Exercise 02

We prove each formula by showing that the derivative of the right–hand side equals the integrand on the left–hand side. This follows from the Fundamental Theorem of Calculus: if $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + C$. Here the constant C is omitted as stated.

1. **Power rule:**
   Let $F(u) = \dfrac{u^{p+1}}{p+1}$, with $p \neq -1$. Then

   $$\frac{d}{du}\left(\frac{u^{p+1}}{p+1}\right) = \frac{1}{p+1} \cdot (p+1)u^p = u^p.$$

   Hence $\displaystyle\int u^p\,du = \frac{u^{p+1}}{p+1}$.

2. **Integral of $1/u$:**
   For $u > 0$, the derivative of $\ln u$ is $\dfrac{1}{u}$. Therefore

   $$\int \frac{du}{u} = \ln u.$$

   (If $u < 0$, the formula becomes $\ln|u|$.)

3. **Linearity of integration:**
   Let $G(x) = \int u\,dx$ and $H(x) = \int v\,dx$, so that

   $$\frac{dG}{dx} = u, \qquad \frac{dH}{dx} = v.$$

   Consider the sum $G(x) + H(x)$. Its derivative is

   $$\frac{d}{dx}\big[G(x) + H(x)\big] = \frac{dG}{dx} + \frac{dH}{dx} = u + v.$$

   Thus $G(x) + H(x)$ is an antiderivative of $u + v$, i.e.

   $$\int (u + v)\,dx = \int u\,dx + \int v\,dx.$$

4. **Integral of cosine:**
   Since $\dfrac{d}{du}(\sin u) = \cos u$, we have immediately

   $$\int \cos u\,du = \sin u.$$

Exercise 03

1. $\int x\sqrt{x^2+1}\,dx$

   Let $u = x^2+1$. Then $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$.
   Substituting:

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{u}\cdot\frac{du}{2} = \frac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \frac{1}{3}u^{3/2} + C$$

   Back substitute $u = x^2+1$:

$$\boxed{\frac{1}{3}(x^2+1)^{3/2} + C}$$

2. $\int x^2 e^{x^3}\cos(e^{x^3})\,dx$

   Let $u = e^{x^3}$. Then $du = e^{x^3}\cdot 3x^2\,dx = 3x^2 e^{x^3}\,dx$,
   so $x^2 e^{x^3}\,dx = \frac{du}{3}$.
   Substituting:

$$\int x^2 e^{x^3}\cos(e^{x^3})\,dx = \int \cos u\cdot\frac{du}{3} = \frac{1}{3}\int \cos u\,du$$
$$= \frac{1}{3}\sin u + C$$

   Back substitute $u = e^{x^3}$:

$$\boxed{\frac{1}{3}\sin(e^{x^3}) + C}$$

3. $\int \frac{1-\cos x}{x-\sin x}\,dx$

   Let $u = x - \sin x$. Then $du = (1-\cos x)\,dx$.
   Substituting:

$$\int \frac{1-\cos x}{x-\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C$$

   Back substitute $u = x - \sin x$:

$$\boxed{\ln|x-\sin x| + C}$$

4. $\int \frac{x^3}{\sqrt{1-x^4}}\,dx$

   Let $u = 1-x^4$. Then $du = -4x^3\,dx$, so $x^3\,dx = -\frac{du}{4}$.
   Substituting:

$$\int \frac{x^3}{\sqrt{1-x^4}}\,dx = \int \frac{1}{\sqrt{u}}\cdot\left(-\frac{du}{4}\right) = -\frac{1}{4}\int u^{-1/2}\,du$$
$$= -\frac{1}{4}\cdot 2u^{1/2} + C = -\frac{1}{2}\sqrt{u} + C$$

   Back substitute $u = 1-x^4$:

$$\boxed{-\frac{1}{2}\sqrt{1-x^4} + C}$$

5. $\int e^x \sin(e^x)\,dx$

Let $u = e^x$. Then $du = e^x\,dx$.

Substituting:

$$\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C$$

Back substitute $u = e^x$:

$$\boxed{-\cos(e^x) + C}$$

Exercise 4: Integration by Parts

(a) Start with the product rule for differentiation:

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrate both sides with respect to x:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx.$$

The left-hand side is simply uv (up to an additive constant, which we omit as usual). Therefore,

$$uv = \int u\,dv + \int v\,du,$$

where we have written $dv = \frac{dv}{dx}dx$ and $du = \frac{du}{dx}dx$. Rearranging gives the integration by parts formula:

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

(b) To evaluate $\int xe^{2x}\,dx$ using integration by parts, we choose:

$$u = x \quad \Rightarrow \quad du = dx,$$

$$dv = e^{2x}\,dx \quad \Rightarrow \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

Applying the formula:

$$\int xe^{2x}\,dx = uv - \int v\,du = x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx.$$

Now compute the remaining integral:

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

Thus,

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2} \cdot \frac{1}{2}e^{2x} + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

So the final result is:

$$\boxed{\int xe^{2x}\,dx = \frac{1}{4}e^{2x}(2x - 1) + C}.$$

Exercise 5: Definite Integrals Using Substitution

(1) Consider $\displaystyle\int_0^1 \frac{3x^2}{x^3+1}\,dx$.

Use the substitution $u = x^3 + 1$. Then $du = 3x^2\,dx$. Change the limits: when $x = 0$, $u = 1$; when $x = 1$, $u = 2$.

The integral becomes:

$$\int_0^1 \frac{3x^2}{x^3+1}\,dx = \int_{u=1}^{u=2} \frac{1}{u}\,du = [\ln|u|]_1^2 = \ln 2 - \ln 1 = \ln 2.$$

Thus,

$$\boxed{\int_0^1 \frac{3x^2}{x^3+1}\,dx = \ln 2}.$$

(2) Consider $\displaystyle\int_0^{\pi/4} \sec^2(2x)\,dx$.

Note: A definite integral $\int_a^b f(x)\,dx$ exists if $f(x)$ is continuous on the closed interval $[a, b]$. Here, $f(x) = \sec^2(2x)$ is not continuous at $x = \pi/4$ because $\sec(2x) = 1/\cos(2x)$ and $\cos(\pi/2) = 0$. Therefore, the integral is improper and must be evaluated as a limit.

We evaluate:

$$\int_0^{\pi/4} \sec^2(2x)\,dx = \lim_{b\to\pi/4^-} \int_0^b \sec^2(2x)\,dx.$$

Use the substitution $u = 2x$, so $du = 2\,dx$ and $dx = du/2$. When $x = 0$, $u = 0$; when $x = b$, $u = 2b$.

Then,

$$\int_0^b \sec^2(2x)\,dx = \int_0^{2b} \sec^2 u \cdot \frac{du}{2} = \frac{1}{2}\,[\tan u]_0^{2b} = \frac{1}{2}\tan(2b).$$

Taking the limit as $b \to \pi/4^-$, we have $2b \to \pi/2^-$, and $\tan(2b) \to +\infty$. Therefore,

$$\lim_{b\to\pi/4^-} \frac{1}{2}\tan(2b) = +\infty.$$

Hence, the integral diverges:

$$\boxed{\int_0^{\pi/4} \sec^2(2x)\,dx \text{ diverges}}.$$

Exercise 6: Area Under a Curve

The area under a curve $y = f(x)$ from $x = a$ to $x = b$ is given by the definite integral

$$A = \int_a^b f(x)\,dx,$$

provided $f(x) \geq 0$ on $[a, b]$.

Here, $f(x) = \cos x$ is nonnegative on $[0, \pi/2]$, so the area is:

$$A = \int_0^{\pi/2} \cos x\,dx.$$

The antiderivative of $\cos x$ is $\sin x$. Thus,

$$A = [\sin x]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.$$

Therefore, the area under the curve is:

$$\boxed{1}.$$

Exercise 7: Mixed Substitution and Definite Integral

1. Consider $\int \frac{\ln x}{x}\,dx$.

   Notice that the derivative of $\ln x$ is $\frac{1}{x}$. Thus, we set:

   $$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx.$$

   Substituting, the integral becomes:

   $$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + C.$$

   Replacing u with $\ln x$:

   $$\boxed{\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C}.$$

2. Consider the definite integral $\int_0^1 xe^{-x^2}\,dx$.

   Observe that the derivative of x^2 is $2x$, so we have an $x\,dx$ factor. Set:

   $$u = x^2 \quad \Rightarrow \quad du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{du}{2}.$$

   Change the limits: when $x = 0$, $u = 0$; when $x = 1$, $u = 1$.
   Substitute into the integral:

   $$\int_0^1 xe^{-x^2}\,dx = \int_0^1 e^{-u} \cdot \frac{du}{2} = \frac{1}{2}\int_0^1 e^{-u}\,du.$$

   Compute the integral:

   $$\frac{1}{2}\int_0^1 e^{-u}\,du = \frac{1}{2}\left[-e^{-u}\right]_0^1 = \frac{1}{2}\left(-e^{-1} - (-e^0)\right) = \frac{1}{2}\left(-\frac{1}{e} + 1\right).$$

   Simplify:

   $$\frac{1}{2}\left(1 - \frac{1}{e}\right) = \frac{1}{2} \cdot \frac{e-1}{e} = \frac{e-1}{2e}.$$

   Thus,

   $$\boxed{\int_0^1 xe^{-x^2}\,dx = \frac{e-1}{2e}}.$$

Exercise 8: Integration by Parts with Trigonometric Functions

We use integration by parts: $\int u\,dv = uv - \int v\,du$.
Choose:

$$u = x \quad \Rightarrow \quad du = dx,$$
$$dv = \cos(3x)\,dx \quad \Rightarrow \quad v = \int \cos(3x)\,dx = \frac{1}{3}\sin(3x).$$

Apply the formula:

$$\int x\cos(3x)\,dx = uv - \int v\,du = x \cdot \frac{1}{3}\sin(3x) - \int \frac{1}{3}\sin(3x)\,dx = \frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin(3x)\,dx.$$

Compute the remaining integral:

$$\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C.$$

Thus,

$$\int x\cos(3x)\,dx = \frac{1}{3}x\sin(3x) - \frac{1}{3}\left(-\frac{1}{3}\cos(3x)\right) + C = \frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C.$$

Therefore,

$$\boxed{\int x\cos(3x)\,dx = \frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C}.$$

Exercise 9: Definite Integral of a Rational Function

Use the substitution $u = 1 + \ln x$. Then

$$du = \frac{1}{x}\,dx \quad \Rightarrow \quad \frac{dx}{x} = du.$$

Change the limits of integration:

$$\text{When } x = 1, \quad u = 1 + \ln 1 = 1 + 0 = 1,$$
$$\text{When } x = e, \quad u = 1 + \ln e = 1 + 1 = 2.$$

The integral becomes:

$$\int_1^e \frac{dx}{x(1+\ln x)} = \int_1^2 \frac{1}{u}\,du.$$

Compute the integral:

$$\int_1^2 \frac{1}{u}\,du = [\ln|u|]_1^2 = \ln 2 - \ln 1 = \ln 2.$$

Therefore,

$$\boxed{\int_1^e \frac{dx}{x(1+\ln x)} = \ln 2}.$$

Exercise 10: Area Between Curves

Step 1: Find intersection points. The curves intersect when $x^2 = \sqrt{x}$. Square both sides (since both sides are nonnegative on $[0,1]$):

$$x^4 = x \quad \Rightarrow \quad x^4 - x = 0 \quad \Rightarrow \quad x(x^3 - 1) = 0.$$

Thus, $x = 0$ or $x^3 = 1$ (i.e., $x = 1$). So the intersection points are $x = 0$ and $x = 1$. These coincide with the given limits.

Step 2: Determine which curve is above on $[0,1]$. Take a test point in $(0,1)$, say $x = 0.25$. Then:

$$y = x^2 = (0.25)^2 = 0.0625, \quad y = \sqrt{x} = \sqrt{0.25} = 0.5.$$

Thus, $\sqrt{x} > x^2$ on $(0,1)$. Therefore, $\sqrt{x}$ is the upper curve and x^2 is the lower curve.

Step 3: Set up the integral for the area. The area between the curves from $x = 0$ to $x = 1$ is:

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)dx.$$

Step 4: Compute the integral. Rewrite $\sqrt{x} = x^{1/2}$. Then:

$$\int \left(x^{1/2} - x^2\right)dx = \frac{x^{3/2}}{3/2} - \frac{x^3}{3} = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3.$$

Evaluate from 0 to 1:

$$A = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \left(\frac{2}{3}(1)^{3/2} - \frac{1}{3}(1)^3\right) - 0 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Therefore, the enclosed area is:

$$\boxed{\frac{1}{3}}.$$

Exercise 11: Proof and Application

1. The proof relies on the fact that differentiation and integration are inverse operations. Recall the derivative of the tangent function:

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Therefore, by the Fundamental Theorem of Calculus, the antiderivative of $\sec^2 x$ is $\tan x$ (up to an additive constant). Hence,

$$\boxed{\int \sec^2 x \, dx = \tan x + C}.$$

2. Using the result from part (a), we evaluate the definite integral:

$$\int_0^{\pi/4} \sec^2 x \, dx = [\tan x]_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1 - 0 = 1.$$

Thus,

$$\boxed{\int_0^{\pi/4} \sec^2 x \, dx = 1}.$$

Exercise 12: Integration of Rational Fractions

1. For $I_1 = \int \frac{2x-1}{(x-1)(x-2)} \, dx$, we use partial fractions:

$$\frac{2x-1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Multiply both sides by $(x-1)(x-2)$:

$$2x - 1 = A(x-2) + B(x-1).$$

Using convenient x-values:

$$x = 1: \ 2(1) - 1 = A(1-2) \Rightarrow 1 = -A \Rightarrow A = -1,$$
$$x = 2: \ 2(2) - 1 = B(2-1) \Rightarrow 3 = B \Rightarrow B = 3.$$

Hence,

$$I_1 = \int \left(-\frac{1}{x-1} + \frac{3}{x-2}\right) dx = -\ln|x-1| + 3\ln|x-2| + C.$$

$$\boxed{I_1 = -\ln|x-1| + 3\ln|x-2| + C}.$$

2. For $I_2 = \int \frac{x}{(x+1)(x+3)(x+5)} \, dx$, we decompose:

$$\frac{x}{(x+1)(x+3)(x+5)} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x+5}.$$

Multiply through by the denominator:

$$x = A(x+3)(x+5) + B(x+1)(x+5) + C(x+1)(x+3).$$

Using convenient x-values:

$$x = -1: \ -1 = A(2)(4) = 8A \Rightarrow A = -\frac{1}{8},$$
$$x = -3: \ -3 = B(-2)(2) = -4B \Rightarrow B = \frac{3}{4},$$
$$x = -5: \ -5 = C(-4)(-2) = 8C \Rightarrow C = -\frac{5}{8}.$$

Thus,

$$I_2 = \int \left(-\frac{1}{8(x+1)} + \frac{3}{4(x+3)} - \frac{5}{8(x+5)} \right) dx = -\frac{1}{8} \ln|x+1| + \frac{3}{4} \ln|x+3| - \frac{5}{8} \ln|x+5| + C.$$

$$\boxed{I_2 = -\frac{1}{8} \ln|x+1| + \frac{3}{4} \ln|x+3| - \frac{5}{8} \ln|x+5| + C}.$$

3. For $I_3 = \int \frac{x^5 + x^4 - 8}{x^3 - 4x} \, dx$, first perform polynomial long division because the numerator's degree is higher.

Polynomial long division (Euclidean division):

Divide $x^5 + x^4 - 8$ by $x^3 - 4x$.

Step 1: $\frac{x^5}{x^3} = x^2$. Multiply $(x^3 - 4x)$ by $x^2 : x^5 - 4x^3$.

Subtract: $(x^5 + x^4 - 8) - (x^5 - 4x^3) = x^4 + 4x^3 - 8$.

Step 2: $\frac{x^4}{x^3} = x$. Multiply $(x^3 - 4x)$ by $x : x^4 - 4x^2$.

Subtract: $(x^4 + 4x^3 - 8) - (x^4 - 4x^2) = 4x^3 + 4x^2 - 8$.

Step 3: $\frac{4x^3}{x^3} = 4$. Multiply $(x^3 - 4x)$ by $4 : 4x^3 - 16x$.

Subtract: $(4x^3 + 4x^2 - 8) - (4x^3 - 16x) = 4x^2 + 16x - 8$.

$$\begin{array}{rr|l}
 & x^5 + x^4 - 8 & x^3 \quad -4x \\ \cline{3-3}
- & & x^2 \quad +x \quad +4 \\
 & x^5 - 4x^3 & \\ \cline{1-2}
 & x^4 + 4x^3 - 8 & \\
- & & \\
 & x^4 - 4x^2 & \\ \cline{1-2}
 & 4x^3 + 4x^2 - 8 & \\
- & & \\
 & 4x^3 - 16x & \\ \cline{1-2}
 & 4x^2 + 16x - 8. &
\end{array}$$

Since the remainder degree (2) is less than the divisor degree (3), we stop. Thus:

$$\frac{x^5 + x^4 - 8}{x^3 - 4x} = x^2 + x + 4 + \frac{4x^2 + 16x - 8}{x^3 - 4x}.$$

Therefore,

$$I_3 = \int (x^2 + x + 4) \, dx + \int \frac{4x^2 + 16x - 8}{x^3 - 4x} \, dx.$$

The first integral is:

$$\int (x^2 + x + 4) \, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x.$$

For the second integral, factor the denominator: $x^3 - 4x = x(x-2)(x+2)$. Decompose:

$$\frac{4x^2 + 16x - 8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}.$$

Multiply through:

$$4x^2 + 16x - 8 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2).$$

Using convenient x-values:

$$\begin{aligned} x=0: &\ -8 = A(-2)(2) = -4A \Rightarrow A = 2, \\ x=2: &\ 4(4)+16(2)-8 = 16+32-8 = 40 = B(2)(4) = 8B \Rightarrow B = 5, \\ x=-2: &\ 4(4)+16(-2)-8 = 16-32-8 = -24 = C(-2)(-4) = 8C \Rightarrow C = -3. \end{aligned}$$

Alternatively, by coefficient comparison (as a check): After expanding:

$$4x^2 + 16x - 8 = (A+B+C)x^2 + (2B-2C)x - 4A.$$

Equating coefficients:

$$\begin{aligned} A+B+C &= 4, \\ 2B-2C &= 16 \Rightarrow B - C = 8, \\ -4A &= -8 \Rightarrow A = 2. \end{aligned}$$

With $A=2$, we have $B+C=2$ and $B-C=8$, solving gives $B=5, C=-3$, consistent.

Hence,

$$\int \frac{4x^2+16x-8}{x^3-4x}\,dx = \int \left(\frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2}\right) dx = 2\ln|x| + 5\ln|x-2| - 3\ln|x+2|.$$

Combining all parts:

$$I_3 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + C.$$

$$\boxed{I_3 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + C}.$$