Mathematics Statistics Informatics

Dr: BOUDEBANE Zineb

University of Mila

Institute of Nature and life Sciences

Department of NLS

Email:z.boudebane@center-univ-mila.dz

March, 2026

Table of contents

Objectives

At the end of this course, the learner will be able to :

- ***Identify*** the basic concepts related to real functions of a real variable .
- ***Distinguish*** between real functions of one variable and several variables.
- ***Test*** the nature of numerical series.
- ***Analyze***the statistical data.
- ***Reorganize*** data into parameters and indicators.

Introduction

To study living things, biology utilizes many tools such as computer science, chemistry, statistics, etc. In particular, it uses mathematics, which is becoming increasingly important. This can be explained by the complexity of observed phenomena and the emergence of new fields of study such as ecology, molecular biology, climatology, population dynamics, and more. All of these require highly sophisticated methods for quantifying and interpreting results. Many of these methods were only introduced in the last century.

Alongside the characteristics of description or classification of living entities, contemporary biology is interested in their birth or emergence, their growth or evolution, and their extinction or disappearance. To conduct these studies, various mathematical models have been developed, which are divided into two categories: those qualified as discrete, which call upon combinatorics, sequences, etc., and those qualified as continuous, which utilize functions of one or more real variables, equations, and differential systems.

The "Statistical Mathematics" course is intended for first-year undergraduate students in the natural and life sciences.

This course is divided into four learning units (chapters). It consists of both a lecture and a tutorial session.

I Prerequisites

To successfully complete this course, you must first have:

- Basic concepts related to functions.
- Some differentiation rules .
- Knowledge of some types and properties of numerical sequences
- A general overview of statistical variable

II Numerical series

1. Specific objectives

By the end of this chapter , the student will be able to

- ***Identify*** real series and their general form.
- ***Explain*** the algebraic operations on numerical series.
- ***Choose*** the appropriate comparison test for a given series.
- ***Apply*** convergences tests, Riemann , d'Alembert , and Cauchy.

2. Series with Real Terms

Definition

Let (U_n)be a sequence with real numbers. We call the infinite sum

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n\geqslant 0} u_n$$

a ***numerical series***, andu_n is called the ***general term*** of the series.

If we put

$$S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} u_k,$$

then (S_n) is called the ***sequence of partial sums*** of the series $\sum_{n\geqslant 0} u_n$.

Example

- The harmonic series:

$$\sum_{n\geqslant 1} \frac{1}{n}.$$

- The geometric series :

$$\sum_{n\geqslant 0} ar^n = a + ar + \cdots + ar^n$$

Definition

A series with real terms $\sum_{n\geq 0} u_n$ is said to ***be convergent*** if the sequence of partial sums

$(S_n)_{n\geq 0}$

converges to a limit S, called the ***sum of the series.***

$$S = \lim_{n\to+\infty} S_n = \sum_{n\geq 0} u_n.$$

✎ *Note*

A numerical series that is not convergent is said to be ***divergent***.

($\lim_{n\to+\infty} S_n = \infty$ or does not exist)

Example

❶Let $\sum_{n\geq 1} u_n$ be the series with general term

$$u_n = \frac{1}{n(n+1)}, \quad n \geq 1.$$

The term u_n can be rewritten as

$$u_n = \frac{1}{n} - \frac{1}{n+1}, \quad n \geq 1.$$

The partial sum is

$$S_n = \sum_{k=1}^{n} u_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Thus,

$$S_n = 1 - \frac{1}{n+1}.$$

Taking the limit,

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore, the series $\sum_{n\geq 1} u_n$ is convergent and its sum is $S = 1$.

❷Let $\sum_{n\geq 1} u_n$ be the series with general term

$$u_n = \ln\left(1 + \frac{1}{n}\right).$$

The term u_n can be rewritten in the form

$$u_n = \ln\left(1 + \frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n).$$

Then

$$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n)$$
$$= \ln(n+1) - \ln(1).$$

Since$\ln(1) = 0$, we obtain

$$S_n = \ln(n+1).$$

Thus,

$$\lim_{n\to+\infty} S_n = +\infty.$$

Therefore, the series $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right) \mathbf{diverges}$.

Fundamental

If the series $\sum u_n$ is convergent, then

$$\lim_{n\to+\infty} u_n = 0.$$

The receprocal is false :

$$(\lim_{n\to+\infty} u_n = 0) \nRightarrow ((\sum u_n)_n \text{ is convergent}).$$

Note

If $\lim_{n\to+\infty} u_n \neq 0$, then the series $(\sum u_n)_n is \mathbf{divergent}$.

Example

1) $\sum_{n=0}^{+\infty} \frac{n}{n+1}$, we have

$$u_n = \frac{n}{n+1} \xrightarrow[n\to+\infty]{} 1 \neq 0.$$

Then $\sum_{n=0}^{+\infty} \frac{n}{n+1}$ diverges.

2) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$ *we have*

$$u_n = \ln\left(1+\frac{1}{n}\right) \xrightarrow[n\to+\infty]{} 0.$$

and, $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$ is divergent

Fundamental:Particular series

1. ***Riemann serie*** :$\sum_{n=1}^{+\infty} \frac{1}{n^{\alpha}}$, converges iff $\alpha > 1$.

2. ***harmonic serie*** : $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.
3. ***geometric serie*** : $\sum_{n=0}^{+\infty} r^n$,, converges iff $|r| < 1$.

Fundamental:Operations on Numerical Series

Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical series, then

- If $\sum_{n\geq 0} u_n$ converges to S_1 and $\sum_{n\geq 0} v_n$ converges to S_2, then $\sum_{n\geq 0}(u_n + v_n)$ converges to $S_1 + S_2$.
- If $\sum_{n\geq 0} u_n$ converges to S and $a \in \mathbb{R}$, then $\sum_{n\geq 0} au_n$ converges to aS.
- If $\sum_{n\geq 0} u_n$ converges and $\sum_{n\geq 0} v_n$ diverges, then $\sum_{n\geq 0}(u_n + v_n)$ diverges.
- If both series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} u_n$ diverge, nothing can be concluded about the nature of the series $\sum_{n\geq 0}(u_n + v_n)$.

Example

Let the series

$$\sum_{n\geq 0}\left(\frac{1}{5^n} + 2^n\right) \quad \text{and} \quad \sum_{n\geq 0}\left(\frac{1}{5^n} - 2^n\right).$$

Put

$$u_n = \frac{1}{5^n} + 2^n, \qquad v_n = \frac{1}{5^n} - 2^n.$$

Then $u_n + v_n = \frac{2}{5^n}$.

Hence,

$$\sum_{n\geq 0}(u_n + v_n) = \sum_{n\geq 0} \frac{2}{5^n},$$

which is a geometric series with ratio $q = \frac{1}{5} < 1$, and therefore convergent.

and we have also

$$u_n - v_n = 2^{n+1}.$$

Then

$$\sum_{n\geq 0}(u_n - v_n) = \sum_{n\geq 0} 2^{n+1},$$

which is divergent $(\text{geometric serie } q = 2 > 1)$

3. Series with Positive Terms

Definition

A series $\sum u_n$ is called a ***series with positive terms*** if

$$u_n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

3.1. Comparison Test

Fundamental:Comparison Rule

Let $(\sum u_n)_n$ and $(\sum v_n)_n$ be two series with positive terms.

Assume that there exists $n_0 \in \mathbb{N}$ such that

$u_n \leq v_n \quad \text{for all } n \geq n_0.$

- If $\sum_{n\geqslant 0} v_n$ converges, then $\sum_{n\geqslant 0} u_n$ converges.
- If $\sum_{n\geqslant 0} u_n$ diverges, then $\sum_{n\geqslant 0} v_n$ diverges.

Example

Consider the series $\sum_{n=0}^{+\infty} \sin\left(\frac{1}{2^n}\right)$.

Since $0 < \sin\left(\frac{1}{2^n}\right) \leq \frac{1}{2^n}$, and since $\sum \frac{1}{2^n}$ is a convergent geometric series,

then the given series converges by comparison.

Extra

Let $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ be two series with positive terms, if there exists $a, b > 0$ satisfying

$au_n \leq v_n \leq bu_n$

then $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ are the same nature.

3.2. Tests of convergence

Fundamental:Test of Cauchy

Let $\sum_{n\geqslant 0} u_n$ be a series with strictly positive terms and if

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} (u_n)^{\frac{1}{n}} = \ell$$

exists.

- If $\ell < 1$, then $\sum_{n\geqslant 0} u_n$ is convergent.
- If $\ell > 1$, then $\sum_{n\geqslant 0} u_n$ is divergent.

- If $\ell = 1$, then we can't say any thing.

Example

- $\sum_{n\geqslant 0}\left(\frac{n+5}{2n+1}\right)^n$
  We have

$$\sqrt[n]{u_n} = \frac{n+5}{2n+1} \xrightarrow[n\to+\infty]{} \frac{1}{2} < 1.$$

Then, the serie is converges.

Fundamental:Test of D'Alembert

Let $\sum_{n\geqslant 0} u_n$ be a series with positive terms. If

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \ell$$

exists, then:

If $\ell < 1$, the serie $\sum_{n\geqslant 0} u_n$ converges.

If $\ell > 1$, the serie $\backslash sum_{n\geqslant 0} u_n$ diverges.

Example

- $\sum_{n\geqslant 0}\frac{2^n}{n!}$.

We compute

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2}{n+1} \xrightarrow[n\to+\infty]{} 0 < 1.$$

Therefore, the serie $\sum_{n\geqslant 0}\frac{2^n}{n!}$ is converges.

Fundamental:Riemann's Test

Let $\sum u_n$ be a series with positive terms and $\alpha \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} n^\alpha u_n = \ell.$$

If$\ell = 0$ and $\alpha > 1$, then$\sum u_n$ converges.

If $\ell = +\infty$ and$\alpha \leq 1$, then$\sum u_n$ diverges.

If$\ell \neq 0 \neq +\infty$, then $\sum u_n$ has the same nature as $\sum \frac{1}{n^\alpha}$.

Example

Let $u_n = \dfrac{1}{n^2 \ln n}$.

We have

$$\lim_{n\to+\infty} n^2 u_n = \frac{1}{\ln n} = 0.$$

Hence, the series $\sum_{n\geq 2} \dfrac{1}{n^2 \ln n}$ convergent with $\alpha = 2 \geqslant 1$.

Bibliography

Jean Bouyer, 2000- Méthodes statistiques : médecine-biologie. Ed. Estem.

Gilles Stoltz et Vincent Rivoirard, 2012- Statistique mathématique en action. Ed. Vuibert, Paris, 448p.

Genet, J. et Pupion, G : Analyse moderne, tome 2, Vuibert, 1974.