

# Rigid body Planar kinetics

## Impulse and momentum

The objective of this lecture is to revisit the principle of impulse and momentum and conservation of momentum. It can be applied to problems involving both **linear** and **angular motion**, and demonstrate how to apply them to solve rigid body planar kinetic problems involving **force**, **moment**, **velocity** and **time**. It can also be used to analyze the mechanics of impact.

الهدف من هذه المحاضرة هو إعادة النظر في مبدأ الدفع وكمية الحركة وحفظ كمية الحركة. يمكن تطبيقها على المسائل التي تتضمن الحركة الخطية والزاوية على حد سواء، وتوضيح كيفية تطبيقها لحل مسائل الحركة المستوية للجسم الصلب تتضمن القوة والعزم والسرعة والزمن. كما يمكن استخدامها لتحليل ميكانيكا التصادم.

## Rigid body Planar kinetics

### Impulse and momentum

We will need to deal with moments, orientations, angular accelerations.

This remains the case with impulse momentum methods, where we will add an angular counterpart to impulse as well as an angular counterpart to momentum.



A dent in an automotive fender can be removed using an impulse tool, which delivers a force over a very short time interval.

How can we determine the magnitude of the linear impulse applied to the fender?

Could you analyze a carpenter's hammer striking a nail in the same fashion?



Sure! When a stake is struck by a sledgehammer, a large impulsive force is delivered to the stake and drives it into the ground.

If we know the initial speed of the sledgehammer and the duration of impact, how can we determine the magnitude of the impulsive force delivered to the stake?

# Linear Impulse

- The **linear impulse** of a constant magnitude force will be equal to the magnitude of the force times the duration of the time.  $\vec{L} = (\vec{F})(t)$ .
- For non constant magnitudes, the linear impulse of any force is equal to the integral of the force function over magnitude of the force times the duration of the set time period .  $\vec{L} = \int \vec{F}(t) dt$  .
- These are vector quantities, and the direction of the force will be the direction of the impulse.

# Angular Impulse

- The **angular impulse** of a constant magnitude moment will be equal to the magnitude of the moment times the duration of the time.  $\vec{H} = (\vec{M}) (t)$ .

- For non constant magnitudes, the linear impulse of any force is equal to the integral of the force function over magnitude of the force times the duration of the set time period .

$$\vec{H} = \int \vec{M}(t) dt .$$

- These are vector quantities, and the direction of the force will be the direction of the impulse vector.

# Linear Momentum

- The linear momentum of a body will be the mass of the body times the velocity of that body.

$$m * \vec{v}$$

- Unlike an impulse which needs to occur over <sup>\*</sup>time, the momentum is instantaneous.
- The momentum has a direction as well. The direction of the momentum will be the instantaneous direction of the velocity at that point.

# Angular Momentum

- The **Angular momentum** of a body at any given instant will be the mass moment of inertia of the body times the angular velocity of that body.

$$I * \vec{\omega}$$

- Just as linear momentum, angular momentum has a direction as well. The direction of the angular momentum vector will be simply be the direction of the angular velocity vector.
- We can take the angular momentum about any point, though we will usually take the angular momentum about one of four points.
- The center of mass of the body.
- A fixed axis of rotation.
- The instantaneous center of zero velocity.
- The point of impact in a collision.

# Angular Momentum about a Fixed Axis or the Instant Center .

To find the angular momentum of a body rotating about a permanently fixed axis, or about instant center .

$$I_O * \vec{\omega} \text{ or } I_{IC} * \vec{\omega}$$

Use the parallel axis theorem for this adjustment

$$I_O = I_G + m r^2 \text{ or } I_{IC} = I_G + m r^2$$

# Angular Momentum about any other point

- If we want to take the angular momentum about any other point.

We will include both **rotational** terms and **translational** terms.

- Momentum due **to rotation** will be.

$$\vec{M}_G = I_G * \vec{\omega}$$

A similar expression to 1 can be derived for the angular momentum if we start from the principle of conservation of **angular momentum**,  $\dot{\mathbf{H}}_G = \mathbf{M}_G$ . Here,  $\mathbf{M}_G = I_G \boldsymbol{\omega}$  is the angular momentum about the center of mass, and  $\mathbf{M}_G$  is the moment of all externally applied forces about the center of mass. Integrating between times  $t_1$  and  $t_2$ , we have,

$$(\mathbf{H}_G)_2 - (\mathbf{H}_G)_1 = \int_{t_1}^{t_2} \mathbf{M}_G dt$$

- Angular Momentum due **to translation** will be.

$$\vec{r}_{G/P} \times (m * \vec{v}_G)$$

- Putting in all together we get..

$$I_G * \vec{\omega} + \vec{r}_{G/P} \times (m * \vec{v}_G)$$

# Angular Momentum about any other point

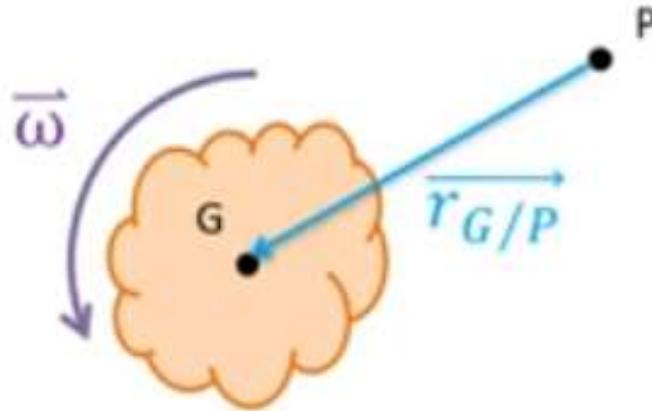
$$I_G * \vec{\omega} + \vec{r}_{G/P} \times (m * \vec{v}_G)$$

Mass Moment of Body About the Center of Mass

Angular Velocity of Body

Displacement Vector from Point of Interest (P) to Center of Mass of Body (G)

Mass of Body



In a similar manner, for rotation about a fixed point O, we can write,

$$(\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 = \int_{t_1}^{t_2} \mathbf{M}_O dt$$

where  $\mathbf{M}_O = \mathbf{I}_O \omega$  the moment of inertia,  $\mathbf{I}_O$ , refers to the fixed point O, and the external moments are with respect to point O.

Finally, if the external applied moment is zero, then we have conservation of angular momentum, which implies  $\omega_2 = \omega_1$

# Impulse and Momentum

We introduced the equations of motion for a two dimensional rigid body.

The **linear momentum** for a system of particles is defined as.  $\mathbf{L} = m \mathbf{v}_G$

where  $m$  is the total mass of the system, and  $\mathbf{v}_G$  is the velocity of the center of mass measured with respect to an inertial reference frame. This equation states that the body's **linear momentum** is a vector quantity having a magnitude  $m v_G$ .

Assuming the mass of the system to be constant, we have that the sum of the external applied forces to the system,  $\mathbf{F}$ , must equal the change in linear momentum,  $\dot{\mathbf{L}} = \mathbf{F}$ , or,

Integrating between times  $t_1$  and  $t_2$ , 
$$\mathbf{L}_2 - \mathbf{L}_1 = \int_{t_1}^{t_2} \mathbf{F} dt$$

This expression is particularly useful when the precise time variation of the applied forces is unknown, when the impulse of the applied forces is **zero**, the momentum is conserved and we have  $(\mathbf{v})_{G2} = (\mathbf{v})_{G1}$ .

## Linear momentum:

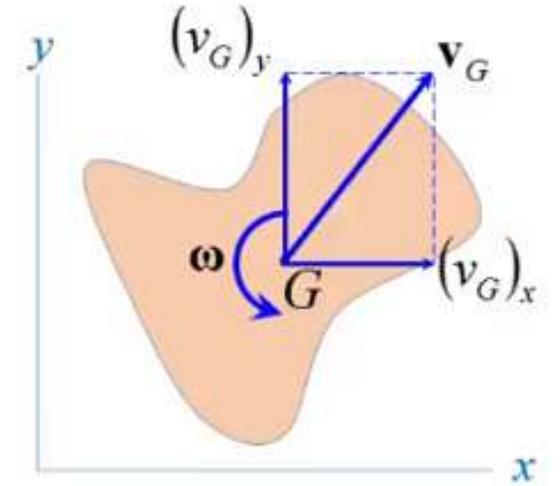
The vector  $m\mathbf{v}$  is called the linear momentum, denoted as  $\mathbf{L}$ . This vector has the same direction as  $\mathbf{v}$ . The linear momentum vector has units of (kg.m)/s. mass is constant.

## Linear impulse :

The integral  $\int \mathbf{F} \cdot dt$  is the linear impulse, denoted  $\mathbf{I}$ . It is **vector quantity** measuring the effect of a force during its time interval of action.  $\mathbf{I}$  acts in the **same direction** as  $\mathbf{F}$  and has units of N.s.

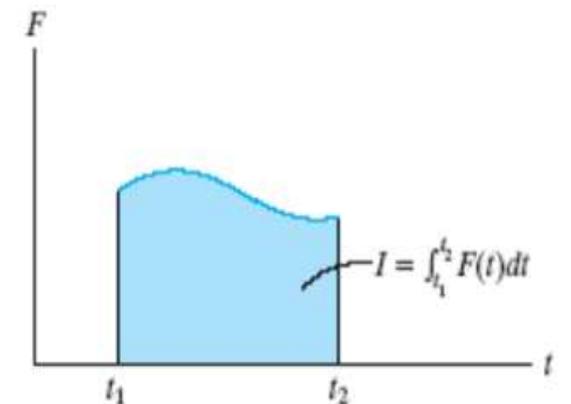
The impulse may be determined by direct integrated. Graphically, it can be represented by the area under the force versus time curve. If  $F$  is constant, then

$$\mathbf{I} = \mathbf{F} (t_1 - t_2) .$$



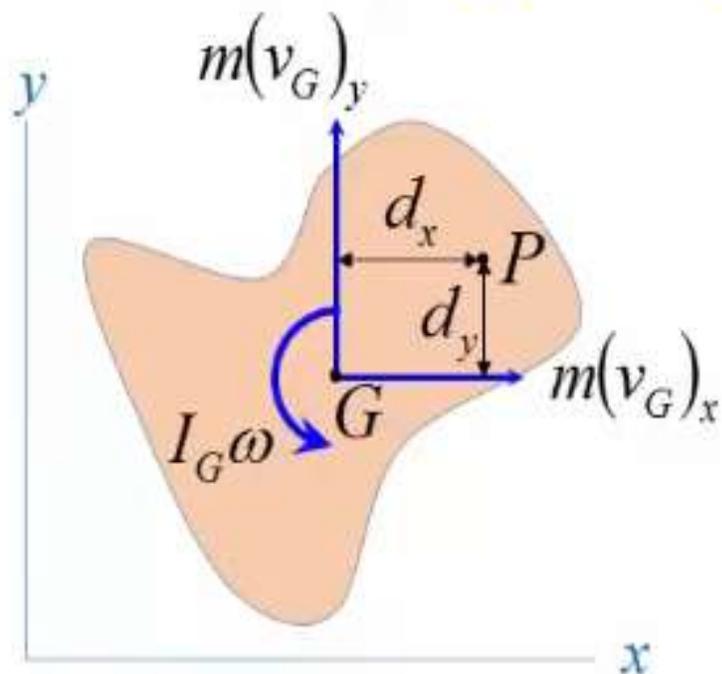
## Linear momentum

$$\mathbf{L} = m\mathbf{v}_G \quad \begin{aligned} L_x &= m(v_G)_x \\ L_y &= m(v_G)_y \end{aligned}$$



Force variable

For **rigid body** planar motion:

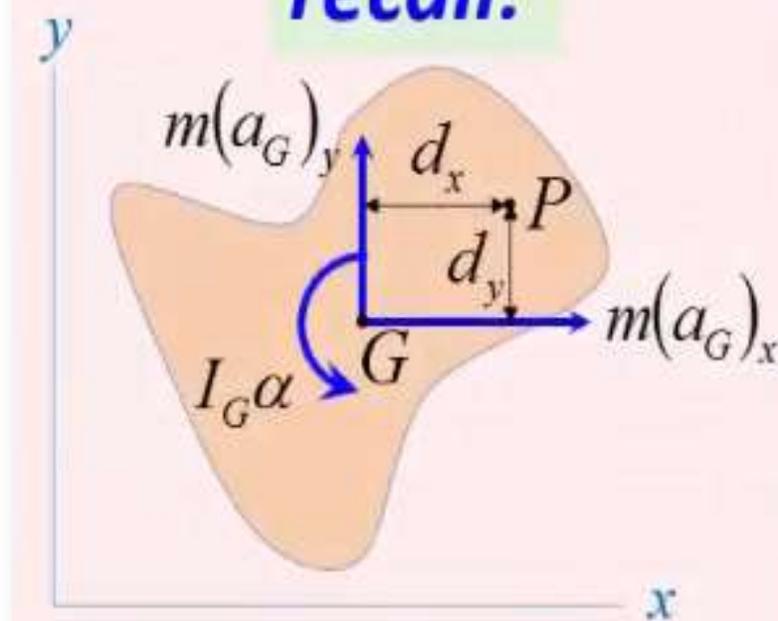


### Angular momentum

about point  $P$ :

$$H_P = m(v_G)_x \cdot d_y - m(v_G)_y \cdot d_x + I_G \omega$$

**recall:**



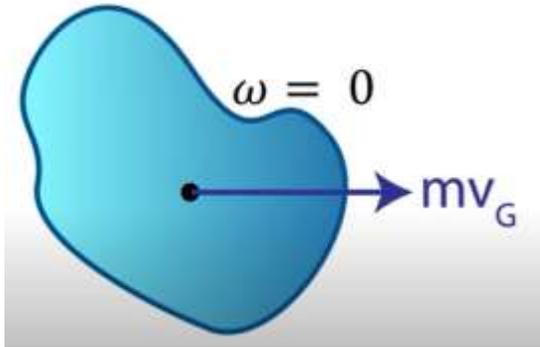
$$\begin{aligned} \sum (\mathcal{M}_k)_P &= m(a_G)_x \cdot d_y \\ &\quad - m(a_G)_y \cdot d_x \\ &\quad + I_G \alpha \end{aligned}$$

kinetic moment about point  $P$

# Momentum

## A—Translation

$$L = mv_G$$
$$H_G = 0$$



## Translation:

$$H_G = 0$$

## Rotation about a fixed axis:

$$H_O = I_O \omega$$
 Try to derive this yourself.

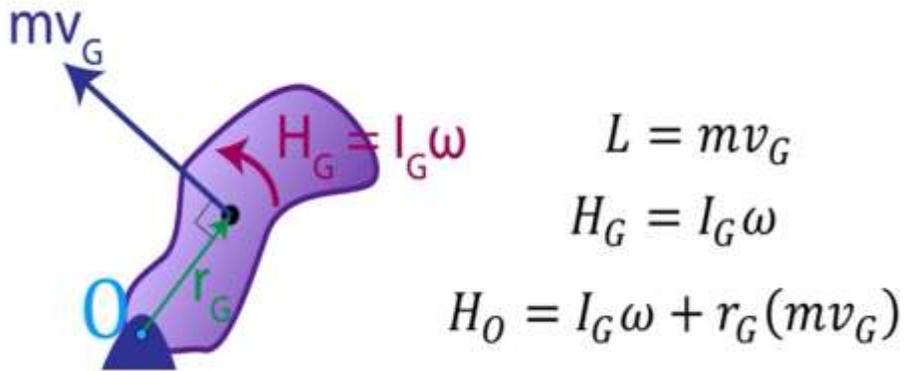
center of rotation

## General plane motion:

$$H_{IC} = I_{IC} \omega$$
 Try to derive this yourself.

instantaneous center of zero velocity

## B—Rotation about a fixed axis

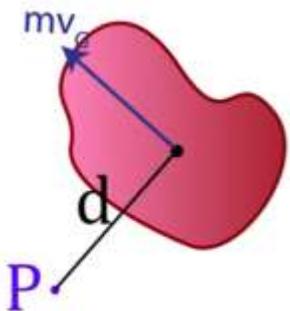


$$L = mv_G$$

$$H_G = I_G \omega$$

$$H_O = I_G \omega + r_G(mv_G)$$

## C— General Plane Motion



$$L = mv_G$$

$$H_G = I_G \omega$$

$$H_P = I_G \omega + d(mv_G)$$

For **rigid body** planar motion:

Principle of **linear** impulse and momentum

$$\mathbf{L}_1 + \sum \mathbf{I}_{1-2} = \mathbf{L}_2$$

Linear  
impulse:

$$\mathbf{I} = \int \mathbf{F} dt$$

Principle of **angular** impulse and momentum

$$(H_P)_1 + \sum \int_{t_1}^{t_2} M_P dt = (H_P)_2$$

## Conservation of momentum

No (or negligible) net **external linear** impulse:

**Linear momentum:**  $\sum m(\mathbf{v}_G)_1 = \sum m(\mathbf{v}_G)_2$

No (or negligible) net **external angular** impulse about point  $P$ :

**Angular momentum:**  $\sum (H_P)_1 = \sum (H_P)_2$

# Principles of impulse and momentum

## 3 scalar equations

$$\begin{cases} m(\mathbf{v}_G)_{x,1} + \sum \int_{t_1}^{t_2} F_x dt = m(\mathbf{v}_G)_{x,2} \\ m(\mathbf{v}_G)_{y,1} + \sum \int_{t_1}^{t_2} F_y dt = m(\mathbf{v}_G)_{y,2} \\ (H_P)_1 + \sum \int_{t_1}^{t_2} M_P dt = (H_P)_2 \end{cases}$$

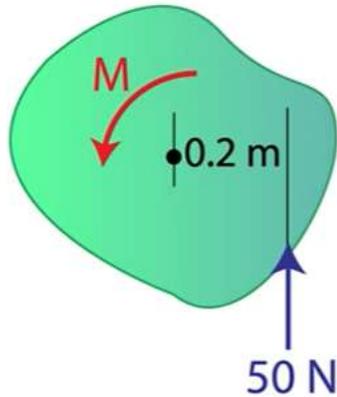
# Impulse

## A–Linear impulse

$$\text{Impulse} = \int_{t_1}^{t_2} F dt$$

## B–Angular impulse

$t = 3s$



$$\text{Angular impulse} = 50(0.2)(3)$$

# Principle of impulse and momentum

$$m(v_{Gx})_1 + \sum \int_{t_1}^{t_2} F_x dt = m(v_{Gx})_2$$

$$m(v_{Gy})_1 + \sum \int_{t_1}^{t_2} F_y dt = m(v_{Gy})_2$$

$$I_G \omega_1 + \sum \int_{t_1}^{t_2} M_G dt = I_G \omega_2$$

Initial Linear  
Momentum

$$\begin{array}{|l} m(v_{Gx})_1 \\ m(v_{Gy})_1 \end{array} + \begin{array}{|l} \sum \int_{t_1}^{t_2} F_x dt \\ \sum \int_{t_1}^{t_2} F_y dt \end{array} = \begin{array}{|l} m(v_{Gx})_2 \\ m(v_{Gy})_2 \end{array}$$

Final Linear  
Momentum

Linear  
Impulses

Angular  
Impulses

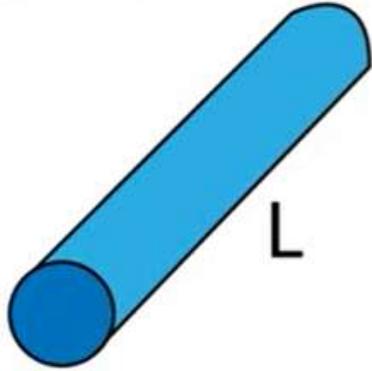
$$\begin{array}{|l} I_G \omega_1 \end{array} + \begin{array}{|l} \sum \int_{t_1}^{t_2} M_G dt \end{array} = \begin{array}{|l} I_G \omega_2 \end{array}$$

Initial  
Angular  
Momentum

Final  
Angular  
Momentum

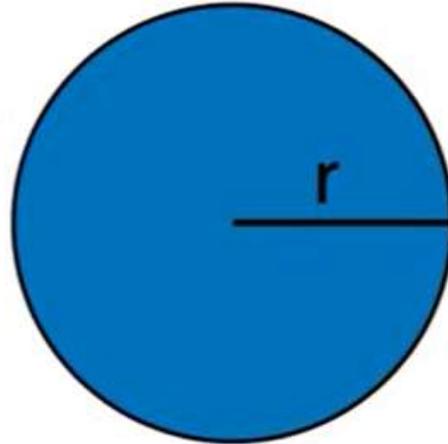
# Mass Moment of Inertia

About Center of  
Uniform Rod



$$I = \frac{1}{12} m(L^2)$$

Uniform Disk



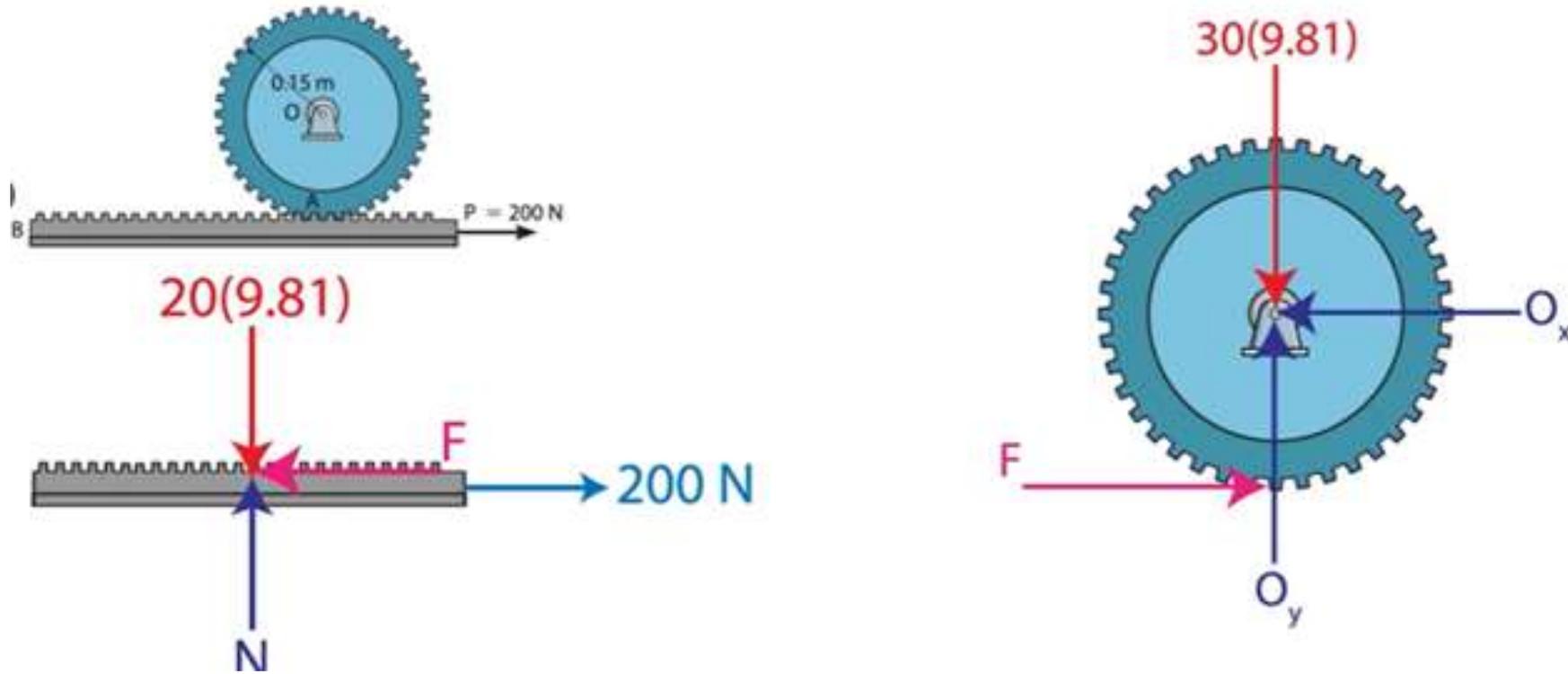
$$I = \frac{1}{2} m(r^2)$$

Radius of  
Gyration

$$I = mk^2$$

## Example 8

The 30 kg gear A has a radius of gyration about its center of mass O of  $k = 125$  mm. If the 20 kg gear rack B is subjected to a force of  $P = 200$  N, determine the time required for the gear to obtain an angular velocity of  $20$  rad/s, starting from rest. The contact surface between the gear rack and the horizontal plane is smooth.



$$\rightarrow^+ m(v_B)_1 + \sum \int_{t_1}^{t_2} F_x dt = m(v_B)_2$$

$$0 + 200(t) - F(t) = 20(3)$$

$$F(t) = 200t - 60$$

$$I_O \omega_1 + \sum \int_{t_1}^{t_2} M_O dt = I_O \omega_2$$

$$0 + F(0.15)(t) = \underline{(0.46875)(20)}$$

$$F(t) = 200t - 60$$

$$0 + F(0.15)(t) = (0.46875)(20)$$

$$F = 102.04 \text{ N}$$

$$t = 0.6125 \text{ s}$$

$$I_O = mk^2$$

$$I_O = (30)(0.125^2)$$

$$I_O = 0.46875 \text{ kg} \cdot \text{m}^2$$