Exercise 13: $\sqrt{2}$ and Sequence Convergence

1. Prove that if $\lim_{n\to\infty} U_{2n} = L$ and $\lim_{n\to\infty} U_{2n+1} = L$, then (U_n) converges to L.

   Proof. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} U_{2n} = L$, there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$,

   $$|U_{2k} - L| < \varepsilon.$$

   Similarly, since $\lim_{n\to\infty} U_{2n+1} = L$, there exists $N_2 \in \mathbb{N}$ such that for all $k > N_2$,

   $$|U_{2k+1} - L| < \varepsilon.$$

   Define $N = \max\{2N_1, 2N_2 + 1\}$. Now take any $n > N$. We consider two cases:

   - If n is even, write $n = 2k$. Then $k > N_1$ (because $n > 2N_1$), so $|U_n - L| = |U_{2k} - L| < \varepsilon$.
   - If n is odd, write $n = 2k + 1$. Then $k > N_2$ (because $n > 2N_2 + 1$ implies $k > N_2$), hence $|U_n - L| = |U_{2k+1} - L| < \varepsilon$.

   Thus for every $n > N$, we have $|U_n - L| < \varepsilon$, which proves that $\lim_{n\to\infty} U_n = L$. □

2. Given $U_1 = 1$ and

   $$U_{n+1} = 1 + \frac{1}{1 + U_n},$$

   compute the first eight terms of (U_n). Then use part (1) to show that $\lim_{n\to\infty} U_n = \sqrt{2}$. This gives the continued fraction expansion

   $$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\cdots}}.$$

Solution. **Computation of the first eight terms:**

$$
\begin{aligned}
U_1 &= 1, \\
U_2 &= 1 + \frac{1}{1+1} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5, \\
U_3 &= 1 + \frac{1}{1+\frac{3}{2}} = 1 + \frac{2}{5} = \frac{7}{5} = 1.4, \\
U_4 &= 1 + \frac{1}{1+\frac{7}{5}} = 1 + \frac{5}{12} = \frac{17}{12} \approx 1.4166667, \\
U_5 &= 1 + \frac{1}{1+\frac{17}{12}} = 1 + \frac{12}{29} = \frac{41}{29} \approx 1.4137931, \\
U_6 &= 1 + \frac{1}{1+\frac{41}{29}} = 1 + \frac{29}{70} = \frac{99}{70} \approx 1.4142857, \\
U_7 &= 1 + \frac{1}{1+\frac{99}{70}} = 1 + \frac{70}{169} = \frac{239}{169} \approx 1.4142012, \\
U_8 &= 1 + \frac{1}{1+\frac{239}{169}} = 1 + \frac{169}{408} = \frac{577}{408} \approx 1.4142157.
\end{aligned}
$$

Step 1: Preliminary properties of f. Define $f(x) = 1 + \dfrac{1}{1+x}$ for $x > 0$. Then $U_{n+1} = f(U_n)$. Observe that:

$$f'(x) = -\frac{1}{(1+x)^2} < 0 \quad \text{for all } x > 0,$$

so f is strictly decreasing. Also note that

$$f(\sqrt{2}) = 1 + \frac{1}{1+\sqrt{2}} = 1 + (\sqrt{2} - 1) = \sqrt{2},$$

since $\frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$ (multiply numerator and denominator by $\sqrt{2} - 1$).

Step 2: Alternating bounds. We prove by induction that for all $n \geq 1$,

$$U_{2n} > \sqrt{2} > U_{2n+1}. \tag{*}$$

Base case ($n = 1$): $U_2 = 1.5 > \sqrt{2} \approx 1.4142 > U_3 = 1.4$.

Inductive step: Assume $U_{2n} > \sqrt{2} > U_{2n+1}$. Since f is decreasing,

$$f(U_{2n}) < f(\sqrt{2}) < f(U_{2n+1}).$$

But $f(U_{2n}) = U_{2n+1}$, $f(\sqrt{2}) = \sqrt{2}$, and $f(U_{2n+1}) = U_{2n+2}$. Hence

$$U_{2n+1} < \sqrt{2} < U_{2n+2},$$

which is exactly (*) with n replaced by $n+1$. This completes the induction.

Step 3: Monotonicity of subsequences. We prove by induction that:

(a) (U_{2n}) is strictly decreasing.

(b) (U_{2n+1}) is strictly increasing.

Base case: $U_2 = 1.5 > U_4 \approx 1.4167$, and $U_3 = 1.4 < U_5 \approx 1.4138$.

Inductive step: Assume $U_{2n} > U_{2n+2}$ and $U_{2n+1} < U_{2n+3}$. Since f is decreasing and $U_{2n+1} < U_{2n+3}$, we have

$$U_{2n+2} = f(U_{2n+1}) > f(U_{2n+3}) = U_{2n+4},$$

which proves $U_{2n+2} > U_{2n+4}$. Similarly, since $U_{2n+2} > U_{2n+4}$ and f is decreasing,

$$U_{2n+3} = f(U_{2n+2}) < f(U_{2n+4}) = U_{2n+5},$$

which proves $U_{2n+3} < U_{2n+5}$.

Thus both statements hold by induction.

Step 4: Convergence of subsequences. From (*), (U_{2n}) is decreasing and bounded below by $\sqrt{2}$, so it converges. Let

$$L_e = \lim_{n\to\infty} U_{2n}.$$

Similarly, (U_{2n+1}) is increasing and bounded above by $\sqrt{2}$, so it converges. Let

$$L_o = \lim_{n\to\infty} U_{2n+1}.$$

Step 5: Determining the limits. Applying the recurrence to even and odd indices respectively:

$$U_{2n+1} = 1 + \frac{1}{1+U_{2n}},$$
$$U_{2n+2} = 1 + \frac{1}{1+U_{2n+1}}.$$

Taking limits as $n \to \infty$ yields

$$L_o = 1 + \frac{1}{1 + L_e}, \quad L_e = 1 + \frac{1}{1 + L_o}.$$

Substitute the first equation into the second:

$$L_e = 1 + \frac{1}{1 + \left(1 + \frac{1}{1+L_e}\right)} = 1 + \frac{1}{2 + \frac{1}{1+L_e}}$$
$$= 1 + \frac{1 + L_e}{2(1 + L_e) + 1} = 1 + \frac{1 + L_e}{2L_e + 3}.$$

Thus,

$$L_e - 1 = \frac{1 + L_e}{2L_e + 3}.$$

Cross-multiplying:

$$(L_e - 1)(2L_e + 3) = 1 + L_e \quad \Rightarrow \quad 2L_e^2 + 3L_e - 2L_e - 3 = 1 + L_e.$$

Simplifying: $2L_e^2 - 4 = 0$, so $L_e^2 = 2$. Since $U_{2n} > 0$, we have $L_e = \sqrt{2}$. Then from $L_o = 1 + \frac{1}{1+\sqrt{2}} = \sqrt{2}$ as well.

Step 6: Conclusion using part (1). We have shown $\lim_{n\to\infty} U_{2n} = \sqrt{2}$ and $\lim_{n\to\infty} U_{2n+1} = \sqrt{2}$. By part (1), the whole sequence (U_n) converges to $\sqrt{2}$.

Step 7: Continued fraction representation. The recurrence can be rewritten as

$$U_{n+1} = 1 + \frac{1}{1 + U_n} = 1 + \frac{1}{2 + (U_n - 1)}.$$

Since $\lim_{n\to\infty} U_n = \sqrt{2}$, letting $n \to \infty$ gives

$$\sqrt{2} = 1 + \frac{1}{2 + (\sqrt{2} - 1)} = 1 + \frac{1}{1 + \sqrt{2}}.$$

Iterating this identity yields the infinite continued fraction:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}.$$

□

Exercise 14: Fish Population Dynamics

The fish population p_n is modeled by:

$$p_{n+1} = \frac{bp_n}{a + p_n},$$

where $a, b > 0$ and $p_0 > 0$.

1. Show that if (p_n) converges, then its limit is either 0 or $b - a$.

   Proof. Assume that $\lim_{n\to\infty} p_n = L$ exists and is finite. Taking the limit on both sides of the recurrence relation:

   $$L = \frac{bL}{a + L}.$$

   Since $p_n > 0$ for all n, we have $L \geq 0$. If $L = 0$, the equation holds. If $L > 0$, we can multiply both sides by $a + L$ to obtain:

   $$L(a + L) = bL \quad \Rightarrow \quad L(a + L) - bL = 0 \quad \Rightarrow \quad L(L + a - b) = 0.$$

   Since $L > 0$, we must have $L + a - b = 0$, i.e., $L = b - a$. Therefore, the only possible limits are 0 and $b - a$. □

2. Show that $p_{n+1} < \frac{b}{a} p_n$.

   Proof. Since $p_n > 0$ and $a > 0$, we have $a + p_n > a$, hence:

   $$\frac{1}{a + p_n} < \frac{1}{a}.$$

   Multiplying by $bp_n > 0$ gives:

   $$\frac{bp_n}{a + p_n} < \frac{b}{a} p_n.$$

   But the left-hand side is exactly p_{n+1}, so $p_{n+1} < \frac{b}{a} p_n$. □

3. Use part (2) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$.

Proof. If $a > b$, then $r := \frac{b}{a} < 1$. From part (2), we have:

$$p_{n+1} < rp_n.$$

By induction, it follows that:

$$p_n < rp_{n-1} < r^2 p_{n-2} < \cdots < r^n p_0.$$

Thus $0 < p_n < r^n p_0$ for all n. Since $0 < r < 1$, we have $\lim_{n\to\infty} r^n = 0$. By the squeeze theorem, we conclude that $\lim_{n\to\infty} p_n = 0$. □

4. Assume $a < b$. Show that if $0 < p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that $\lim_{n\to\infty} p_n = b - a$.

Proof. We first compute the difference between two consecutive terms:

$$p_{n+1} - p_n = \frac{bp_n}{a + p_n} - p_n = \frac{bp_n - p_n(a + p_n)}{a + p_n} = \frac{p_n(b - a - p_n)}{a + p_n}. \tag{1}$$

Also, we compute the difference with $b - a$:

$$p_{n+1} - (b - a) = \frac{bp_n}{a + p_n} - (b - a) = \frac{bp_n - (b - a)(a + p_n)}{a + p_n} = \frac{a(a - b + p_n)}{a + p_n}. \tag{2}$$

Both denominators are positive since $a > 0$ and $p_n > 0$.

Case 1: $0 < p_0 < b - a$.

We prove by induction that $0 < p_n < b - a$ for all n.

Base case: $0 < p_0 < b - a$ holds by assumption.

Inductive step: Assume $0 < p_n < b - a$. From (2), since $p_n < b - a$, we have $a - b + p_n < 0$, so the numerator of (2) is negative, hence $p_{n+1} - (b - a) < 0$, i.e., $p_{n+1} < b - a$. Also, since $p_n > 0$, $p_{n+1} > 0$. Thus $0 < p_{n+1} < b - a$.

Moreover, from (1), since $p_n < b - a$, we have $b - a - p_n > 0$, so $p_{n+1} - p_n > 0$. Hence the sequence is strictly increasing.

Therefore, $\{p_n\}$ is increasing and bounded above by $b - a$, so it converges. By part (1), the limit must be either 0 or $b - a$. Since the sequence is positive and increasing, the limit cannot be 0 (unless it starts at 0, but $p_0 > 0$). Thus $\lim_{n\to\infty} p_n = b - a$.

Case 2: $p_0 > b - a$.

We prove by induction that $p_n > b - a$ for all n.

Base case: $p_0 > b - a$ holds by assumption.

Inductive step: Assume $p_n > b - a$. From (2), since $p_n > b - a$, we have $a - b + p_n > 0$, so the numerator of (2) is positive, hence $p_{n+1} - (b - a) > 0$, i.e., $p_{n+1} > b - a$.

Moreover, from (1), since $p_n > b - a$, we have $b - a - p_n < 0$, so $p_{n+1} - p_n < 0$. Hence the sequence is strictly decreasing.

Therefore, $\{p_n\}$ is decreasing and bounded below by $b - a$, so it converges. Again, by part (1), the limit must be either 0 or $b - a$. Since $p_n > b - a$ for all n, the limit cannot be 0 (it must be at least $b - a > 0$ because $a < b$). Thus $\lim_{n\to\infty} p_n = b - a$.

In both cases, we conclude that when $a < b$, regardless of the initial condition $p_0 > 0$, the sequence converges to $b - a$. $\square$