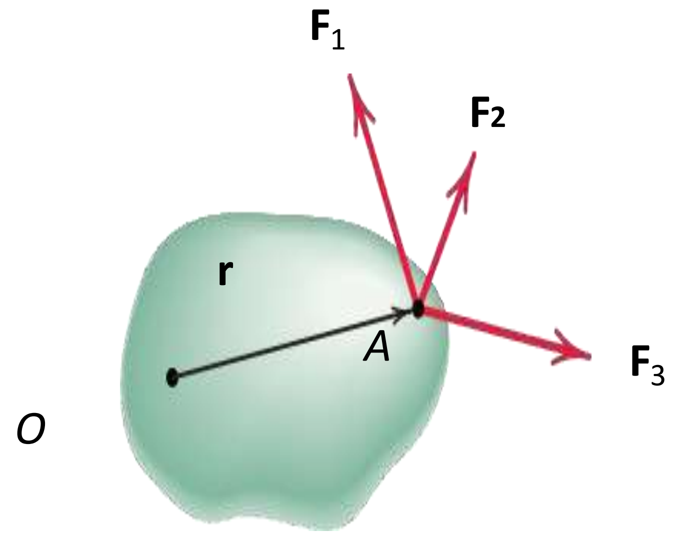
Varignon's Theorem in Three Dimensions

The theorem is easily extended to three dimensions. Figure 1 shows a system of concurrent forces F_1, F_2, F_3, ... The sum of the moments about ***O*** of these forces is

$$\mathbf{r} \times \mathbf{F_1} + \mathbf{r} \times \mathbf{F_2} + \mathbf{r} \times \mathbf{F_3} + \cdots = \mathbf{r} \times (\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \cdots) = \mathbf{r} \times \Sigma\mathbf{F}$$

where we have used the distributive law for cross products. Using the symbol $\boldsymbol{M_O}$ to represent the sum of the moments on the left side of the above equation, we have

$$\mathbf{M_O} = \Sigma\,(\mathbf{r} \times \mathbf{F}) = \mathbf{r} \times \mathbf{R}$$



Resultant Moment of a System of Forces.

The system of forces **F**1, **F**2, **F**3 . . . acting on a rigid body in **Fig. *a***, we may move each of them in turn to the arbitrary point *O*, provided we also introduce a couple for each force transferred. Thus, for example, we may move force **F**1 to *O*, provided we introduce the couple **M**1 = **r**1 × **F**1, where **r**1 is a vector from *O* to any point on the line of action of **F**1. When all forces are shifted to *O* in this manner, we have a system of concurrent forces at *O* and a system of couple vectors, as represented in part *b* of the figure. The concurrent forces may then be added vectorially to produce a resultant force **R**, and the couples may also be added to produce a resultant couple **M**, **Fig.*c***. The general force system, then, is reduced to

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma(\mathbf{r} \times \mathbf{F})$$

One of the two Golden Jubilee Bridges in London, England, adjacent to the Hungerford Bridge. The cables of this bridge exert a three-dimensional system of concentrated forces on each bridge tower.

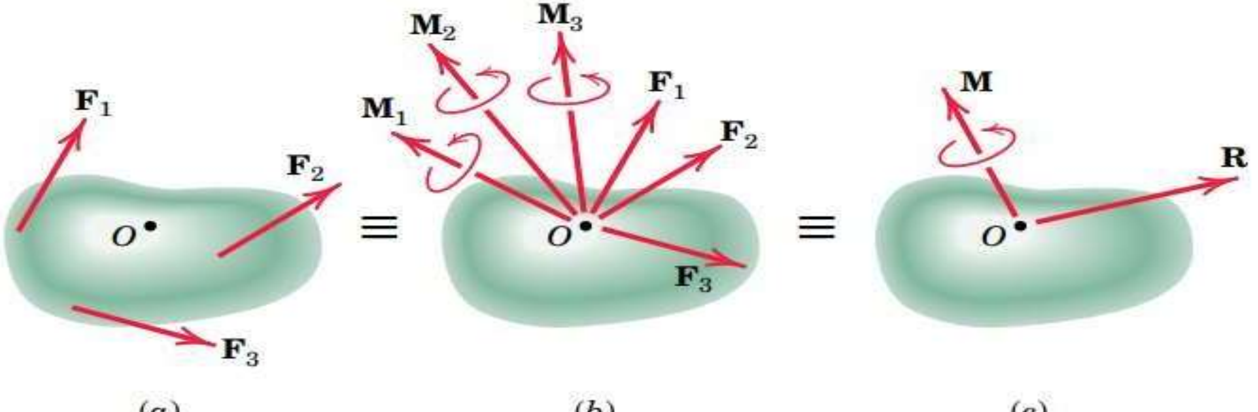
The couple vectors are shown through point *O*, but because they are free vectors, they may be represented in any parallel positions. The magnitudes of the resultants and their components are

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$\mathbf{M}_x = \Sigma(\mathbf{r} \times \mathbf{F})_x \qquad \mathbf{M}_y = \Sigma(\mathbf{r} \times \mathbf{F})_y \qquad \mathbf{M}_z = \Sigma(\mathbf{r} \times \mathbf{F})_z$$

$$M = \sqrt{M_x{}^2 + M_y{}^2 + M_z{}^2}$$



Equilibrium of a System of Forces

Introduction

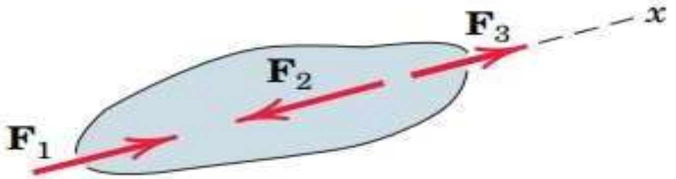
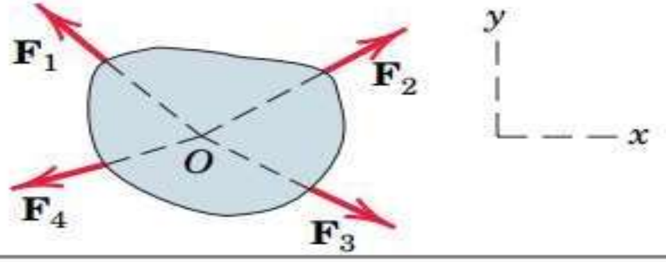
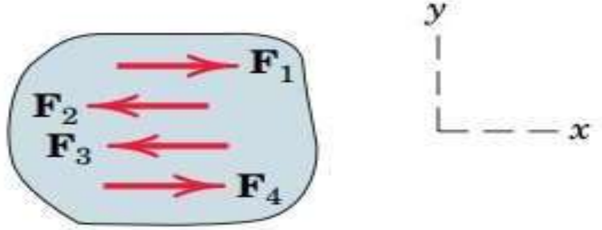
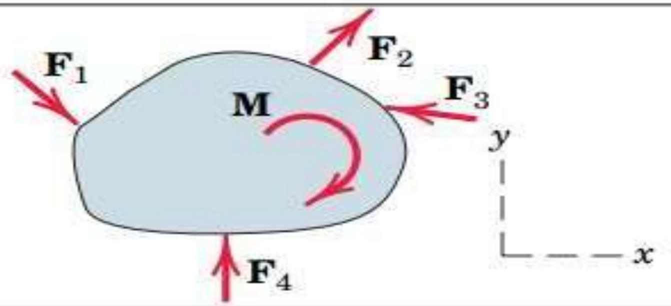
a system of forces is **zero**, the body will remain at **rest** or **move** with constant velocity, if it was already moving with constant velocity, i.e., its acceleration will be zero; and if the moment is also zero. We will discuss a special case that arises when the resultant **force** and **moment** turn out to be **zero**, then there will not be any rotational motion. Such a condition is called **static equilibrium**.

.

$$\mathbf{R} = \Sigma \mathbf{F} = 0 \qquad \mathbf{M} = \Sigma \mathbf{M} = 0$$

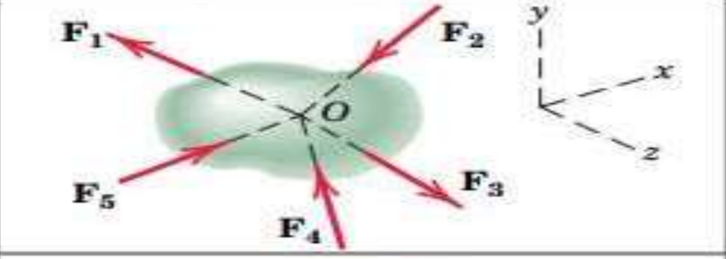
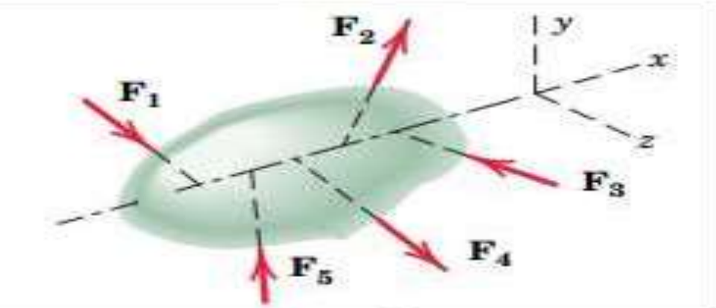
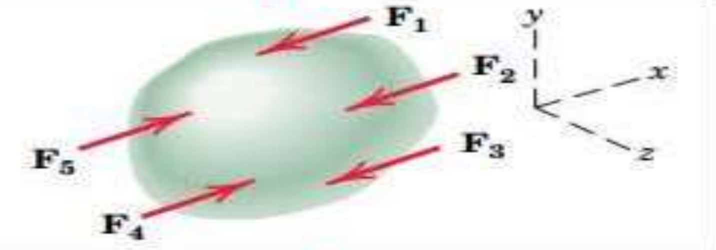
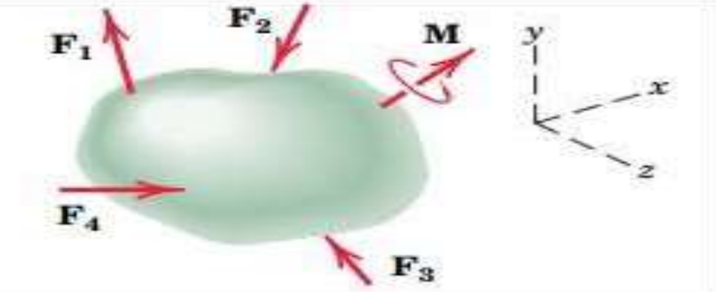
Categories of Equilibrium

The categories of force systems acting on bodies in two-dimensional equilibrium are summarized in a table 1.

CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

Categories of Equilibrium

The categories of force systems acting on bodies in three-dimensional equilibrium are summarized in a table 2.

CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\quad \Sigma M_y = 0$ $\Sigma F_y = 0$ $\quad \Sigma M_z = 0$ $\Sigma F_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\quad \Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\quad \Sigma M_x = 0$ $\Sigma F_y = 0$ $\quad \Sigma M_y = 0$ $\Sigma F_z = 0$ $\quad \Sigma M_z = 0$

Free-Body Diagrams

A free-body diagram is a sketch of the outlined shape of the body, which represents it as being isolated or “ free” from its surroundings, i.e., a “ free body.” On this sketch it is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when the equations of equilibrium are applied. To construct a free-body diagram for a rigid body or any group of bodies considered as a single system, the following steps should be performed:

- **Draw Outlined Shape.**
- **Show All Forces and Couple Moments.**
- **Identify Each Loading and Give Dimensions.**
- **Note:** the weight W of the body locates at the center of gravity.

- Example.1
- Draw free-body diagrams for the following cases:
- (i) A block or a ball resting on a smooth horizontal plane

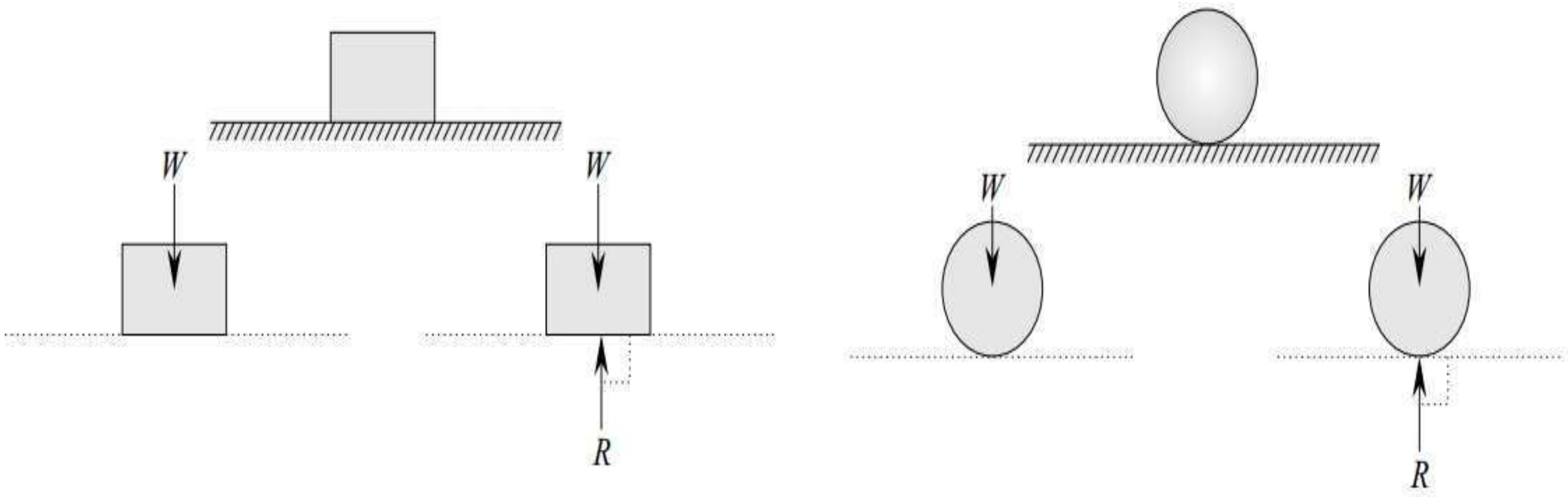


Fig.1. A ball resting.

(ii) A block on a smooth inclined plane is restrained from moving downwards by a string attached to

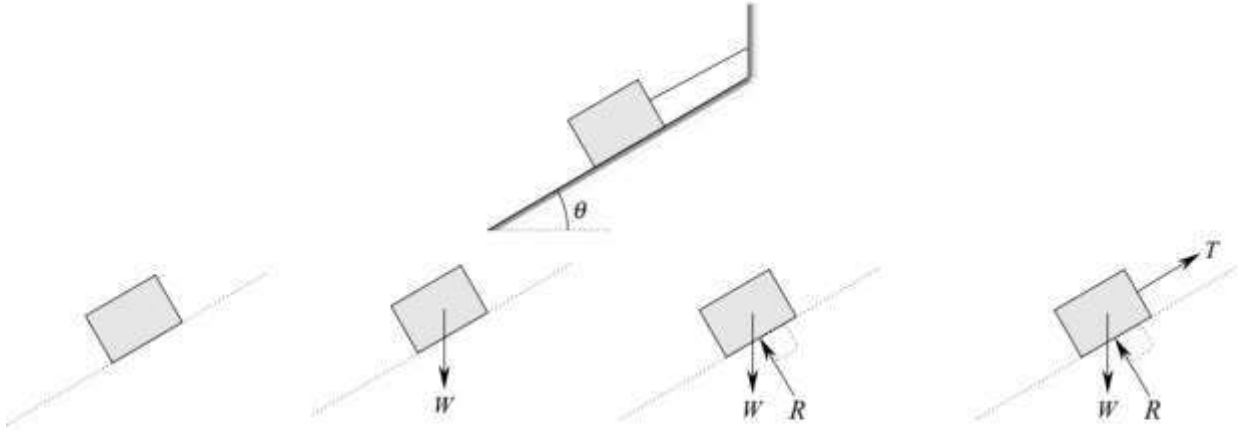


Fig.2. A block on a smooth inclined plane.

(iii) A sphere on a smooth inclined plane is restrained from moving downwards by a vertical plane

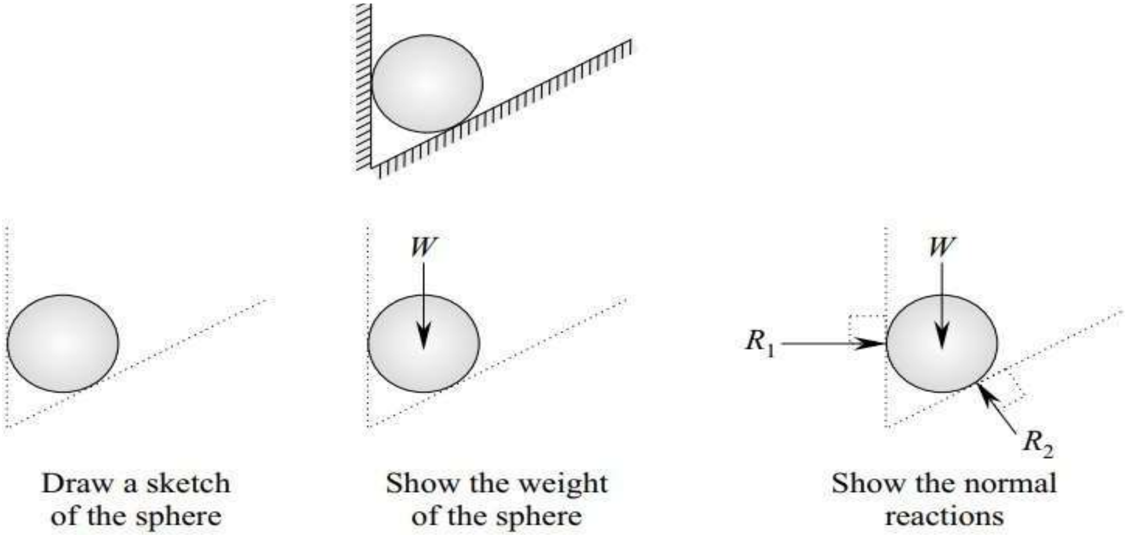
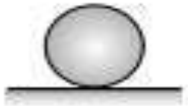
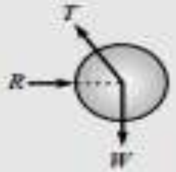
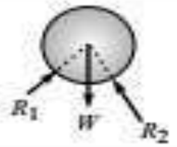
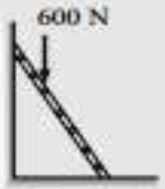
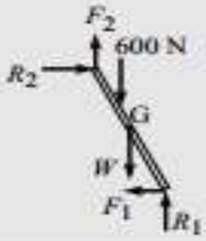
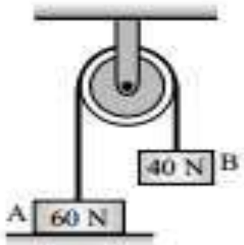
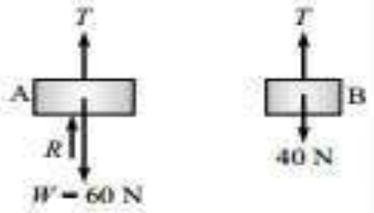
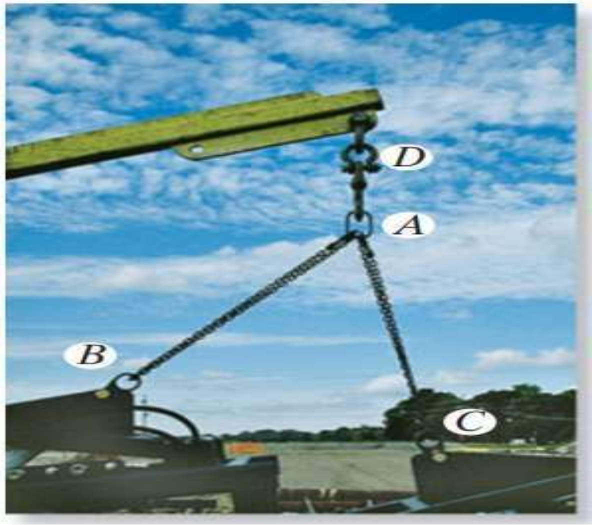


Fig.3. A sphere on a smooth inclined plane.

Table.1. Free-body diagram of some Important cases.

No.	Equilibrium System	The Body	FBD
1.		**Ball** (Resting on a horizontal surface)	
2.		**Ball** (Hanging by a string and supported by a wall)	
3.		**Ball** (Resting between two inclined walls)	
4.		**Loaded Ladder** (Kept against a wall) {Friction exists at both the wall and floor}	
5.		**Block** A (Resting on floor and pulled up by pulley arrangement)	



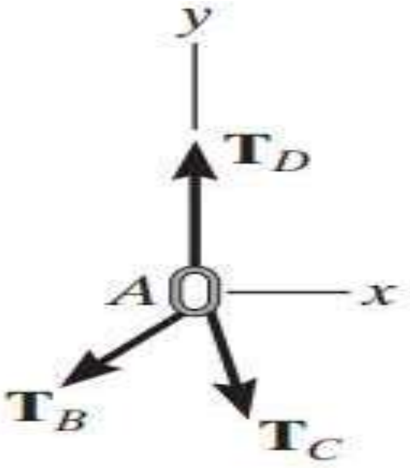


Fig.4. The process of drawing a FBD is illustrated by the sequence shown.

- The chains exert three forces on the ring at ***A***, as shown on its **free-body diagram**. The ring will not move, or will move with constant velocity, provided the summation of these forces along the *x* and along the *y* axis is zero. If one of the three forces is known, the magnitudes of the other two forces can be obtained from the two equations of equilibrium.
- Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider the various **types of reactions that occur at supports** and points of contact between bodies subjected to coplanar force systems. As a general rule,

Types of Support

A structure beam may have the following types of supports.

1. **Roller Support** In this case, end of the beam rests on some sliding surface, rollers or any flat surface like a
smooth masonry wall. It is free to roll or move in the horizontal direction.
2. **Hinged Support** It resists Vertical and horizontal displacements, but allows rotation. It has two reactions- vertical (V) and (H), as shows in Fig.4.
3. **Fixed Support** A fixed support is built monolithically with the wall, this connection is so stiff that it does not allow any kind of displacement (neither translation nor rotation).

This support has three reactions, a vertical reaction (V), a horizontal reaction (H) and moments (M).
A fixed support is also known as ***a built- in support***.

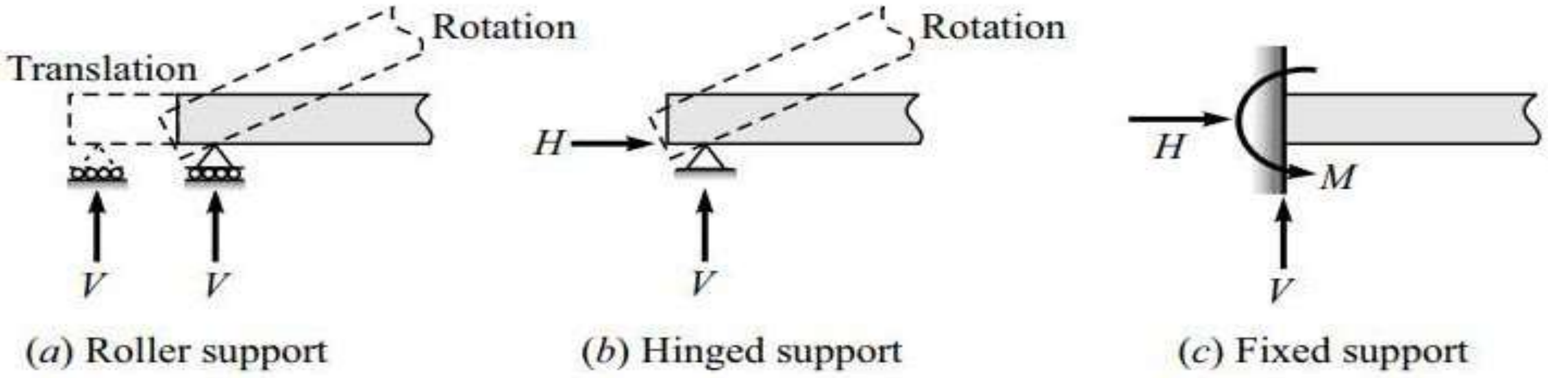
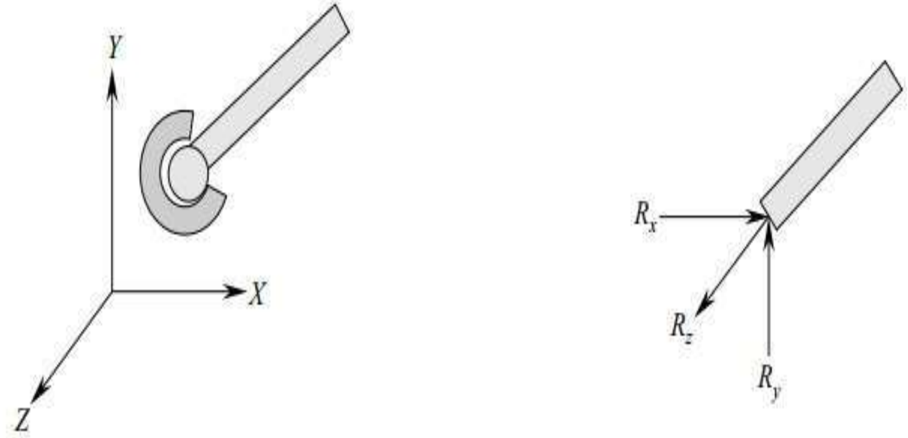


Fig.4. Types of supports.

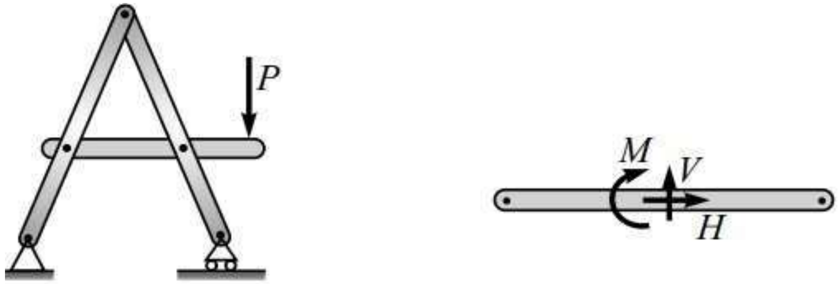
Fig.4.

4. **Ball-and-socket joint** This type of support is same as the hinge support in three dimensions. Here, the body is restrained from moving in all three directions, but is free to rotate in any direction. Hence, reactions R_x, R_y and R_z acton the free-body diagram of the body



5. Frames

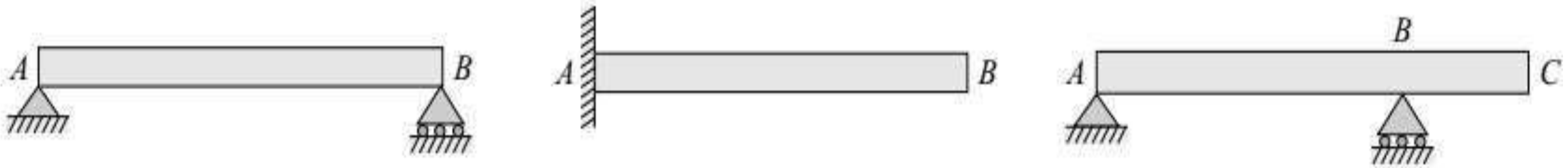
Frames are the structures made of interconnected inextensible members. They are designed to support applied loads and moments. A simple A frame is shown in Fig.5.



(a) A-Frame (b) Multi-force-member of frame

Beams

Beams and Types of Beams One of the structural member that we come across in this sections is a beam. It is a horizontal structural member that is designed to resist forces transverse to its axis, It is held in position by various supports. Depending upon the nature of the supports, beams can be classified as follows.



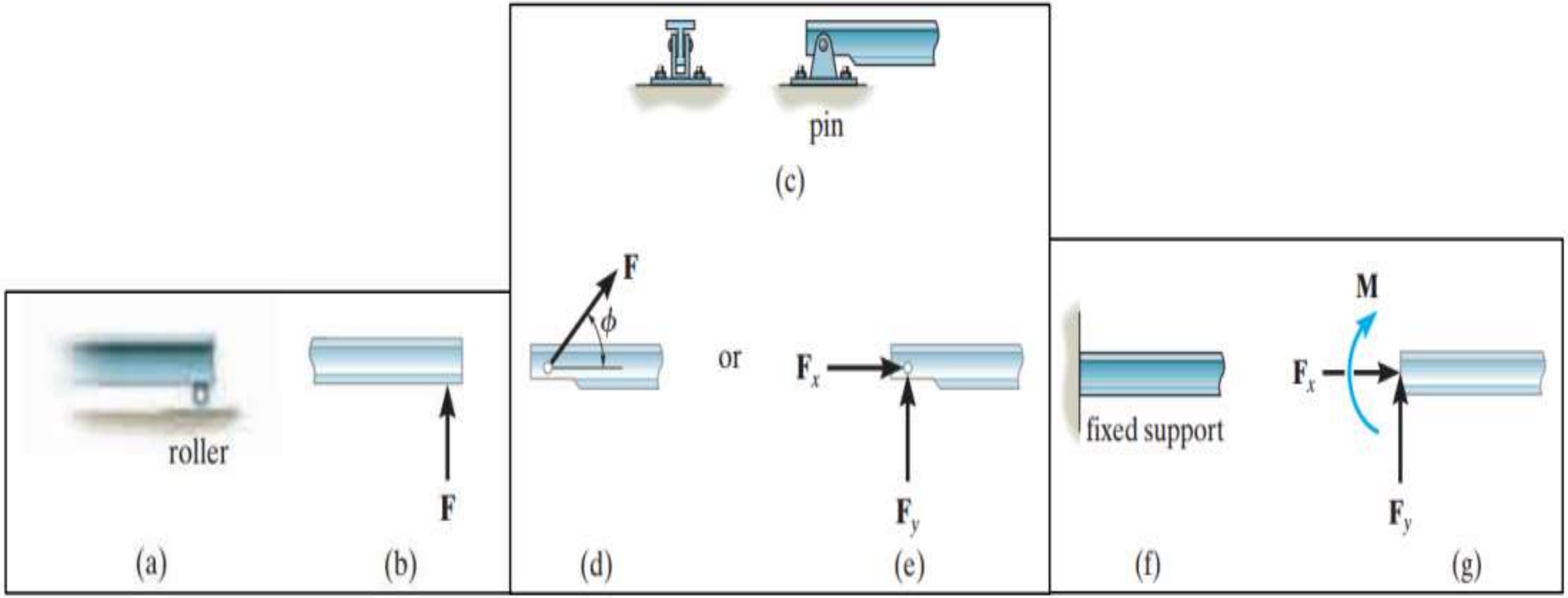
(a) Simply supported beam (b) Cantilever beam (c) Overhanging beam

Simply supported Beam Simply supported beam that is supported by a hinge one end and a roller at the other end.

Cantilever beam is a beam that is supported by a built-in or fixed support at one end and the other end free.

Overhanging beam The beam is simply supported at A and B but it also projects beyond the support to the point C, which is a free end.

Main Support Reactions



Roller Pin (hinge) Fixed

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link		One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  roller or pin in confined smooth slot		One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod		One unknown. The reaction is a force which acts perpendicular to the rod.

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge	 or 	Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support	 or 	Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Equilibrium Conditions

A body is in equilibrium if all forces and moments applied to it are in balance. These requirements are contained in the vector equations of equilibrium, which in **two dimensions** may be written in scalar form as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0$$

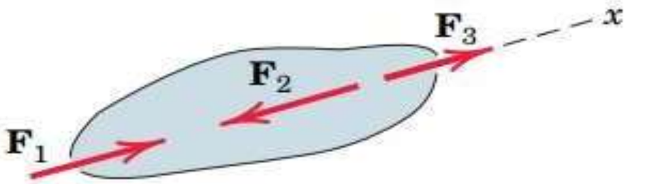
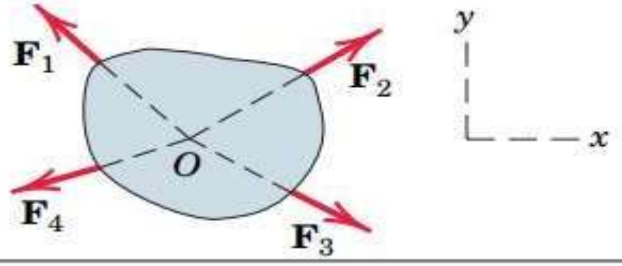
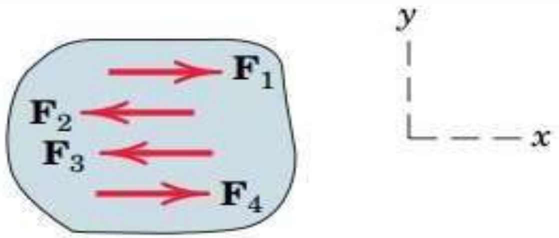
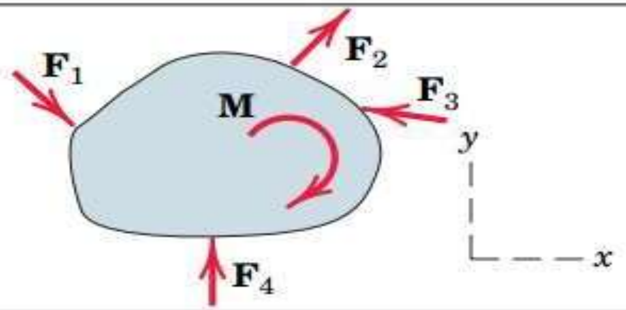
$$\sum Fx = 0$$

$$\sum Fy = 0$$

$$\sum M = 0$$

Categories of Equilibrium

The categories of force systems acting on bodies in two-dimensional equilibrium are summarized in a table.1

CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	**Free-Body Diagram**	**Independent Equations**
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

Equilibrium of Concurrent Forces in Space

- For a concurrent force system in space, expressing the resultant in terms of orthogonal components of individual forces, the equilibrium condition can be stated as

$$\vec{R} = (\sum F_x)\vec{i} + (\sum F_y)\vec{j} + (\sum F_z)\vec{k} = \vec{O}$$

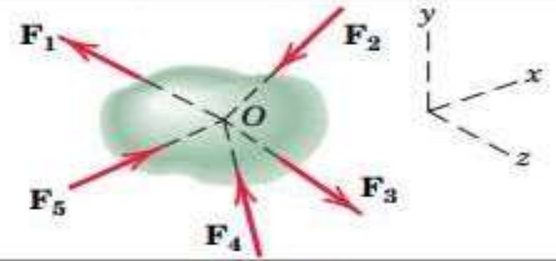
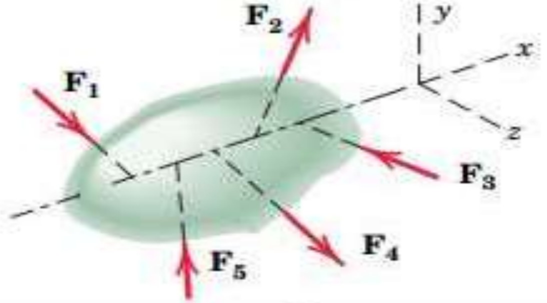
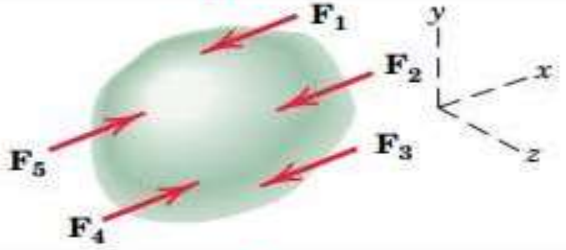
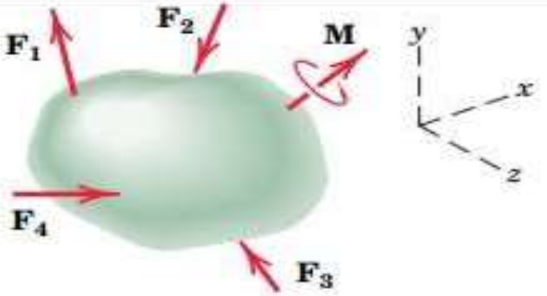
Equilibrium of Non-Concurrent Forces in Space

- If all the forces acting on the body are non-concurrent in space, the necessary and sufficient conditions for equilibrium will be same as that for coplanar non-concurrent forces with the addition of the Z-component for , and X and Y components for .

$$\Sigma \mathbf{F} = \mathbf{0} \quad \text{or} \quad \begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma F_z = 0 \end{cases}$$

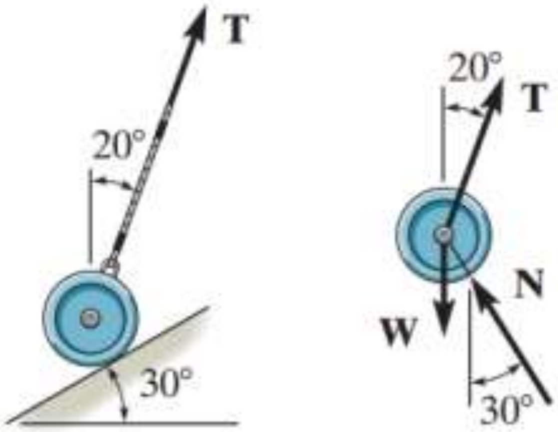
$$\Sigma \mathbf{M} = \mathbf{0} \quad \text{or} \quad \begin{cases} \Sigma M_x = 0 \\ \Sigma M_y = 0 \\ \Sigma M_z = 0 \end{cases}$$

The categories of force systems acting on bodies in three-dimensional equilibrium are summarized in a table.2

CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma F_y = 0$ $\Sigma M_z = 0$ $\Sigma F_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_x = 0$ $\Sigma F_y = 0$ $\Sigma M_y = 0$ $\Sigma F_z = 0$ $\Sigma M_z = 0$

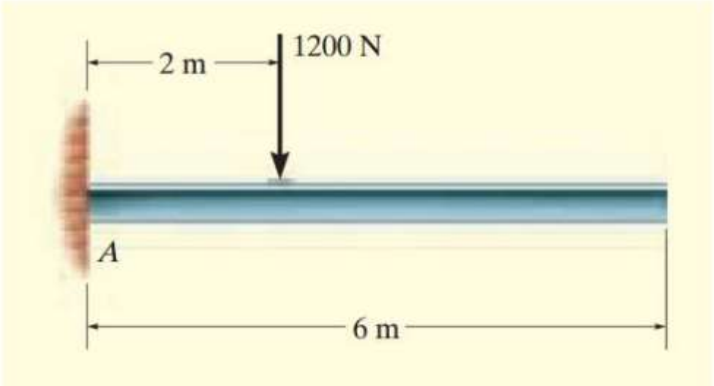
Example 1

Smooth Contact: If an object rests on a smooth surface, then the surface will exert a force on the object that is normal to the surface at the point of contact.

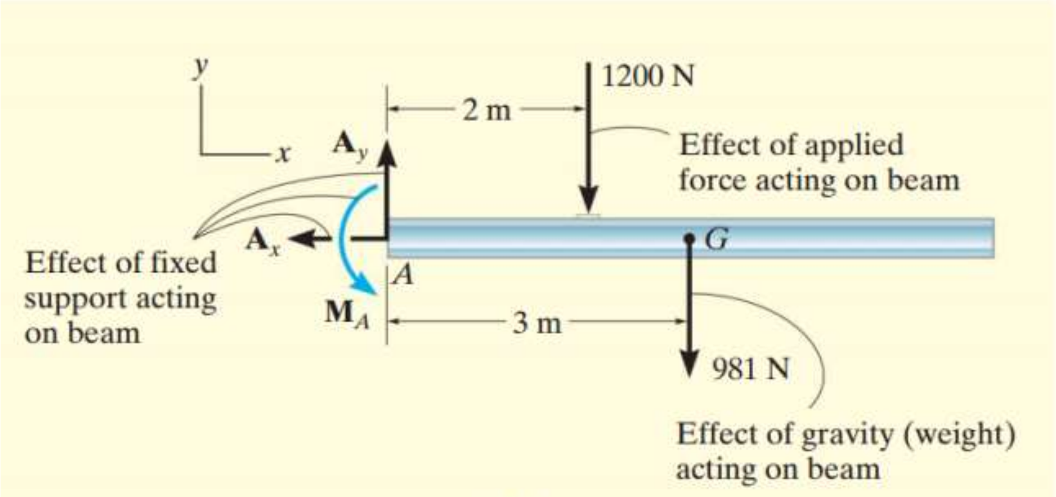


Example 2:

Draw the free-body diagram of the uniform beam shown in Fig. below. The beam has a mass of 100 kg.



(a)

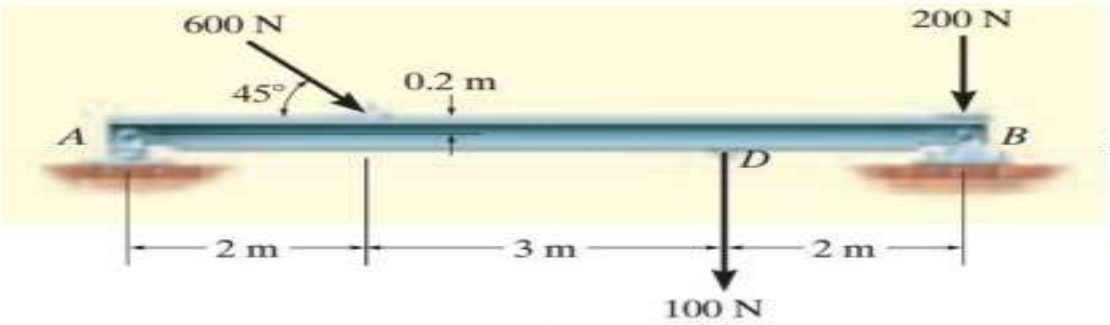


(b)

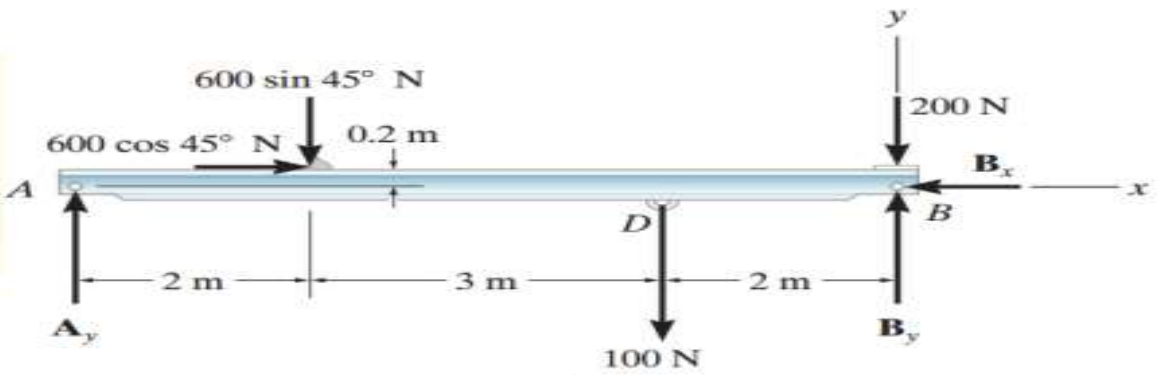
Example:

Determine the horizontal and vertical components of reaction on the beam caused by the pin at B and the roller at A as shown in Fig. a. below. Neglect the weight of the beam.

Solution:



(a)



(b)

Equations of Equilibrium. Summing forces in the x direction yields

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 600 \cos 45° \text{ N} - B_x = 0$$

$$B_x = 424 \text{ N} \qquad \textit{Ans.}$$

A direct solution for $\mathbf{A}_y$ can be obtained by applying the moment equation $\Sigma M_B = 0$ about point B.

$$\circlearrowleft + \Sigma M_B = 0; \qquad 100 \text{ N}(2 \text{ m}) + (600 \sin 45° \text{ N})(5 \text{ m})$$

$$- (600 \cos 45° \text{ N})(0.2 \text{ m}) - A_y(7 \text{ m}) = 0$$

$$A_y = 319 \text{ N} \qquad \textit{Ans.}$$

Summing forces in the y direction, using this result, gives

$$+\uparrow \Sigma F_y = 0; \qquad 319 \text{ N} - 600 \sin 45° \text{ N} - 100 \text{ N} - 200 \text{ N} + B_y = 0$$

$$B_y = 405 \text{ N} \qquad \textit{Ans.}$$