Planar Kinetics of a Rigid Body

- Force and Acceleration
- Work and Energy
- Impulse and Momentum

Centre of Gravity

- Centre de gravity of a body is the point of through which the whole weight of the body acts.
- A body is having only one centre of gravity for all positions of the body.
- It is represented by C.G or simply G.
- Centroid . The point at which the total area

of a plane figure like (rectangle, square, triangle, quatrilateral, circle) is assumed to be concentrated, is known as the centroid of that area.

Centre de gravity and the centroid are at the same point.

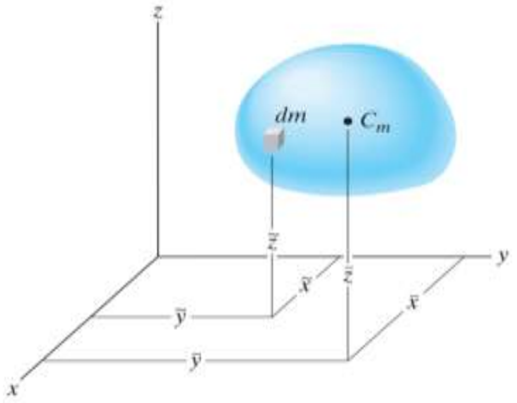


Figure 4.2: Center of Gravity

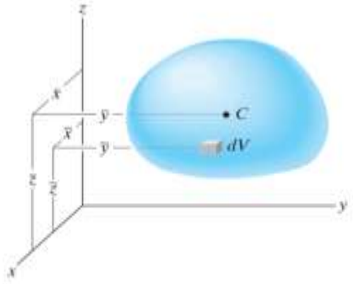


Figure 4.3: Centroid of a Volume

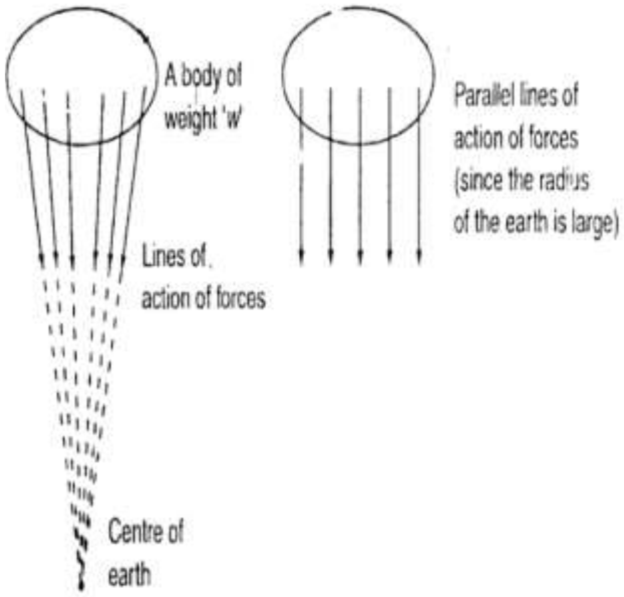


Figure 4.1: Center of Earth.

5.3. CENTROID OR CENTRE OF GRAVITY OF SIMPLE PLANE FIGURES

(i) The centre of gravity (C.G.) of a uniform rod lies at its middle point.

(ii) The centre of gravity of a triangle lies at the point where the three medians* of the triangle meet.

(iii) The centre of gravity of a rectangle or of a parallelogram is at the point, where its diagonal meet each other. It is also the point of intersection of the lines joining the middle points of the opposite sides.

(iv) The centre of gravity of a circle is at its centre.

5.4. CENTROID (OR CENTRE OF GRAVITY) OF AREAS OF PLANE FIGURES BY THE METHOD OF MOMENTS

Fig. 5.1 shows a plane figure of total area A whose centre of gravity is to be determined. Let this area A is composed of a number of small areas a_1, a_2, a_3, a_4, ... etc.

$$\therefore \quad A = a_1 + a_2 + a_3 + a_4 + \dots$$

Let x_1 = The distance of the C.G. of the area a_1 from axis OY

x_2 = The distance of the C.G. of the area a_2 from axis OY

x_3 = The distance of the C.G. of the area a_3 from axis OY

x_4 = The distance of the C.G. of the area a_4 from axis OY

and so on.

The moments of all small areas about the axis OY

$$= a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \dots \qquad \dots(i)$$

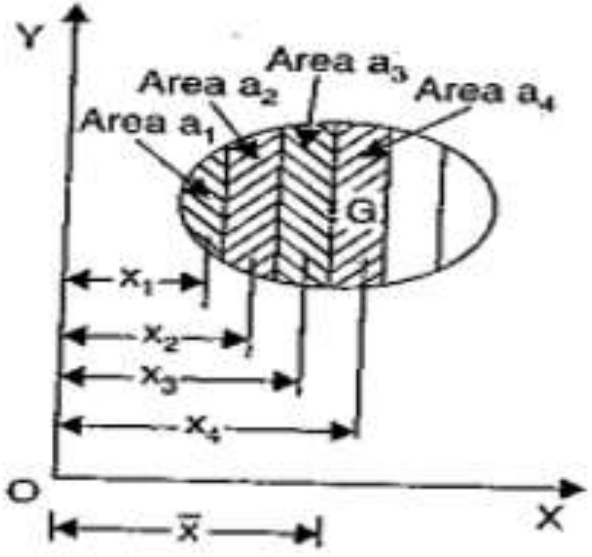


Fig. 5.1

Let G is the centre of gravity of the total area A whose distance from the axis OY is $\bar{x}$.

Then moment of total area about $OY = A\bar{x}$ ...(*ii*)

The moments of all small areas about the axis OY must be equal to the moment of total area about the same axis. Hence equating equations (*i*) and (*ii*), we get

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots = A\bar{x}$$

or

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots}{A} \quad \ldots(5.1)$$

where $A = a_1 + a_2 + a_3 + a_4 \ldots$

If we take the moments of the small areas about the axis OX and also the moment of total area about the axis OX, we will get

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + \ldots}{A} \quad \ldots(5.2)$$

where $\bar{y}$ = The distance of G from axis OX

y_1 = The distance of C.G. of the area a_1 from axis OX

y_2, y_3, y_4 = The distance of C.G. of area a_2, a_3, a_4 from axis OX respectively.

5.4.1. Centre of Gravity of Areas of Plane Figures by Integration Method. The equations (5.1) and (5.2) can be written as

$$\bar{x} = \frac{\sum a_i x_i}{\sum a_i} \quad \text{and} \quad \bar{y} = \frac{\sum a_i y_i}{\sum a_i}$$

where $i = 1, 2, 3, 4, \ldots\ldots$

x_i = Distance of C.G. of area a_i from axis OY and

y_i = Distance of C.G. of area a_i from axis OX.

The value of i depends upon the number of small areas. If the small areas are large in number (mathematically speaking infinite in number), then the summations in the above equations can be replaced by integration. Let the small areas are represented by dA instead of 'a', then the above equations are written as :

$$\bar{x} = \frac{\int x^* \, dA}{\int dA} \qquad \text{...(5.2 } A\text{)}$$

and

$$\bar{y} = \frac{\int y^* \, dA}{\int dA} \qquad \text{...(5.2 } B\text{)}$$

where $\int x^* \, dA = \Sigma x_i a_i$

$\int dA = \Sigma a_i$

$\int y^* dA = \Sigma y_i a_i$

Also x^* = Distance of C.G. of area dA from axis OY

y^* = Distance of C.G. of area dA from axis OX.

5.4.2. Centroid (or Centre of Gravity) of a Line. The centre of gravity of a line which may be straight or curve, is obtained by dividing the given line, into a large number of small lengths as shown in Fig. 5.1 (a).

The centre of gravity is obtained by replacing dA by dL in equations (5.2 A) and (5.2 B).

Then these equations become $\bar{x} = \frac{\int x^* \, dL}{\int dL}$ ...(5.2 C)

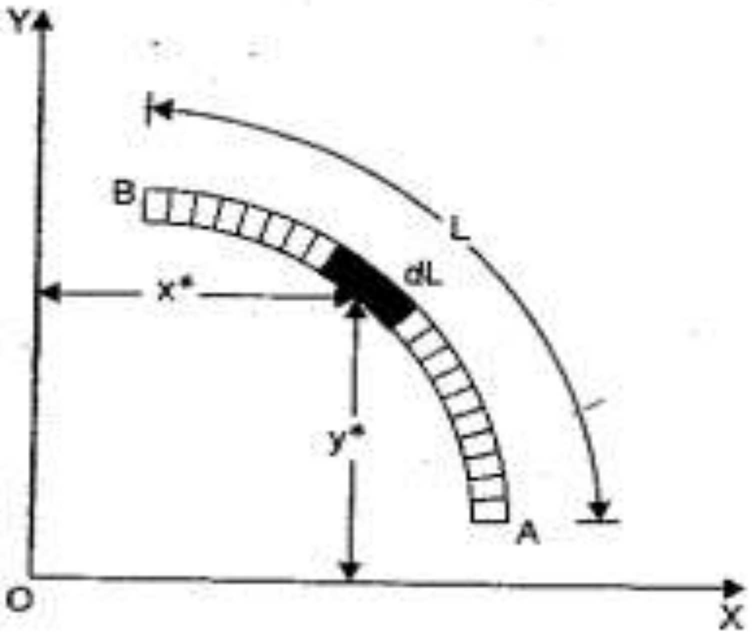


Fig. 5.1 (a)

and
$$\bar{y} = \frac{\int y^* \, dL}{\int dL} \quad \text{...(5.2 D)}$$

where x^* = Distance of C.G. of length dL from y-axis, and

y^* = Distance of C.G. of length dL from x-axis.

If the lines are straight, then the above equations are written as :

$$\bar{x} = \frac{L_1 x_1 + L_2 x_2 + L_3 x_3 + \ldots\ldots}{L_1 + L_2 + L_3 + \ldots\ldots} \quad \text{...(5.2 E)}$$

and
$$\bar{y} = \frac{L_1 y_1 + L_2 y_2 + L_3 y_3 + \ldots\ldots}{L_1 + L_2 + L_3 + \ldots\ldots} \quad \text{...(5.2 F)}$$

Example

Determine the co-ordinates of the C.G. of the shaded area between the parabola y= X2/4 and y=x.

The point A is lying on the straight line as well as on the given parabola. Hence both the above equations holds good for point A. Let the co-ordinates of point A are x, y.

Substituting the value of y from equation (ii) in equation (i), we get

$$x = \frac{x^2}{4} \quad \text{or} \quad 4 = \frac{x^2}{x} = x$$

Substituting the value of $x = 4$, in equation (ii),

$$y = 4$$

Hence the co-ordinates of point A are 4, 4.

Now divide the shaded area into large small areas each of height y and width dx as shown in Fig. 5.8. Then area dA of the strip is given by

$$dA = ydx = (y_1 - y_2)\, dx \qquad \text{...}(iii)$$

where y_1 = Co-ordinate of point D which lies on the straight line OA

y_2 = Co-ordinate of the point E which lies on the parabola OA.

The horizontal co-ordinates of the points D and E are same.

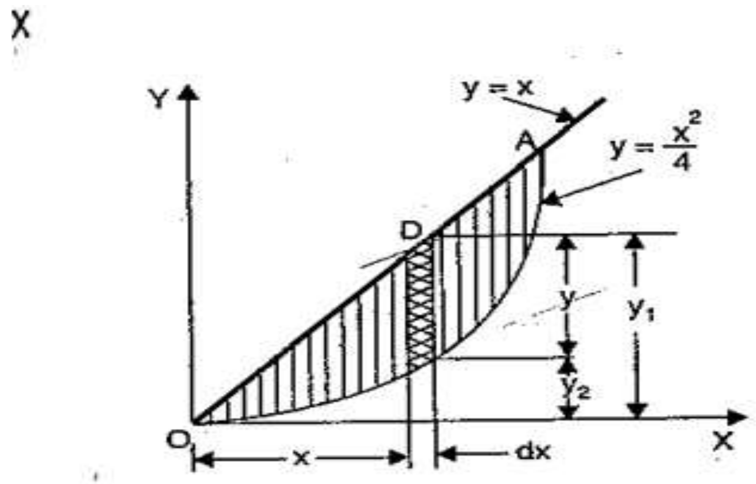


Fig. 5.8

The values of y_1 and y_2 can be obtained in terms of x from equations (*ii*) and (*i*),

$$y_1 = x \quad \text{and} \quad y_2 = \frac{x^2}{4}$$

Substituting these values in equation (*iii*),

$$dA = \left(x - \frac{x^2}{4}\right) dx \qquad \text{...(iv)}$$

The distance of the C.G. for the area dA from y-axis is given by,

$$x^* = x$$

And the distance of the C.G. of the area dA from x-axis is given by,

$$y^* = y_2 + \frac{y}{2} = y_2 + \frac{y_1 - y_2}{2} \qquad (\because \quad y = y_1 - y_2)$$

$$= \frac{2y_2 + y_1 - y_2}{2} = \frac{y_1 + y_2}{2}$$

$$= \frac{x + \frac{x^2}{4}}{2} \qquad \left(\because \quad y_1 = x \text{ and } y_2 = \frac{x^2}{4}\right)$$

$$= \frac{1}{2}\left(x + \frac{x^2}{4}\right) \qquad \text{...(v)}$$

Now let $\bar{x}$ = Distance of C.G. of shaded area of Fig. 5.8 from y-axis

$\bar{y}$ = Distance of C.G. of shaded area of Fig. 5.8 from x-axis.

Now using equation (5.2 A),

$$\bar{x} = \frac{\int x^* \, dA}{\int dA} \qquad \text{where} \quad x^* = x$$

$$dA = \left(x - \frac{x^2}{4}\right) dx \qquad \text{[See equation (iv)]}$$

$$\therefore \quad \int x^* \, dA = \int_0^4 x\left(x - \frac{x^2}{4}\right) dx \qquad (\because \quad x \text{ varies from 0 to 4})$$

$$= \int_0^4 \left(x^2 - \frac{x^3}{4}\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4 \times 4}\right]_0^4$$

$$= \frac{4^3}{3} - \frac{4^4}{4 \times 4} = \frac{64}{3} - 16$$

$$= \frac{64 - 48}{3} = \frac{16}{3}$$

and

$$\int dA = \int_0^4 \left(x - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3 \times 4}\right]_0^4 = \frac{4^2}{2} - \frac{4^3}{3 \times 4}$$

$$= \frac{16}{2} - \frac{16}{3} = \frac{48 - 32}{6} = \frac{16}{6} \qquad \text{...(vi)}$$

$$\therefore \qquad \bar{x} = \frac{\int x^* \, dA}{\int dA} = \frac{\frac{16}{3}}{\frac{16}{6}} = \frac{16}{3} \times \frac{6}{16} = \mathbf{2.} \quad \textbf{Ans.}$$

Now using equation (5.2 B), $\bar{y} = \frac{\int y^* \, dA}{\int dA}$

where $y^* = \frac{1}{2}\left[x + \frac{x^2}{4}\right]$ [From equation (*v*)]

$$dA = \left(x - \frac{x^2}{4}\right) dx$$

$$\therefore \qquad \int y^* \, dA = \int_0^4 \frac{1}{2}\left(x + \frac{x^2}{4}\right)\left(x - \frac{x^2}{4}\right) dx$$

$$= \frac{1}{2}\int_0^4 \left(x^2 - \frac{x^4}{16}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5 \times 16}\right]_0^4$$

$$= \frac{1}{2}\left[\frac{4^3}{3} - \frac{4^5}{5 \times 16}\right] = \frac{1}{2}\left[\frac{64}{3} - \frac{64}{5}\right]$$

$$= \frac{64}{2}\left[\frac{1}{3} - \frac{1}{5}\right] = 32\left(\frac{5-3}{15}\right)$$

$$= 32 \times \frac{2}{15} = \frac{64}{15}$$

and $\int dA = \frac{16}{6}$ [From equation (*vi*)]

$$\therefore \qquad \bar{y} = \frac{\int y^* \, dA}{\int dA} = \frac{\frac{64}{15}}{\frac{16}{6}} = \frac{64}{15} \times \frac{6}{16} = \mathbf{\frac{8}{5}}. \quad \textbf{Ans.}$$

Center of Mass

The centre of mass (***CM***) is the point where the mass-weighted position vectors (moments) relative to the point sum to zero ; the ***CM*** is the mean location of a distribution of mass in space.

Take a system of ***n*** particles, each with mass ***mi*** located at positions *ri* , the position vector of the ***CM*** is defined by:

$$\sum_{i=1}^{n} m_i(\underline{\mathbf{r}}_{\mathrm{i}} - \underline{\mathbf{r}}_{\mathrm{cm}}) = 0$$

Solve for $\underline{\mathbf{r}}_{\mathrm{cm}}$:

$$\underline{\mathbf{r}}_{\mathrm{cm}} = \frac{1}{M}\sum_{i=1}^{n} m_i\,\underline{\mathbf{r}}_{\mathrm{i}}$$

where $M = \sum_{i=1}^{n} m_i$

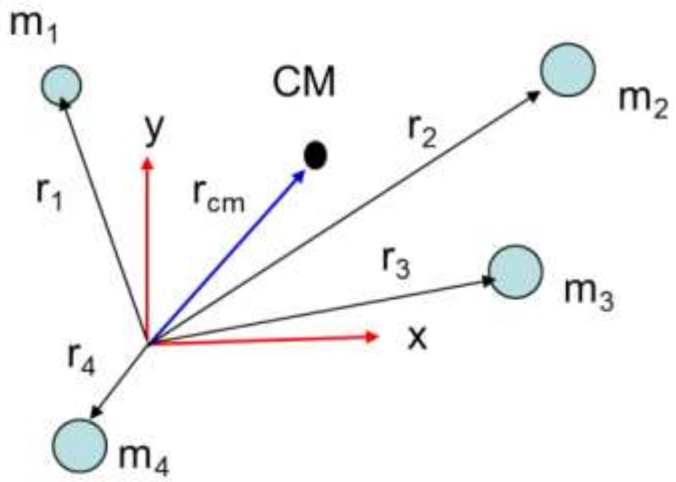
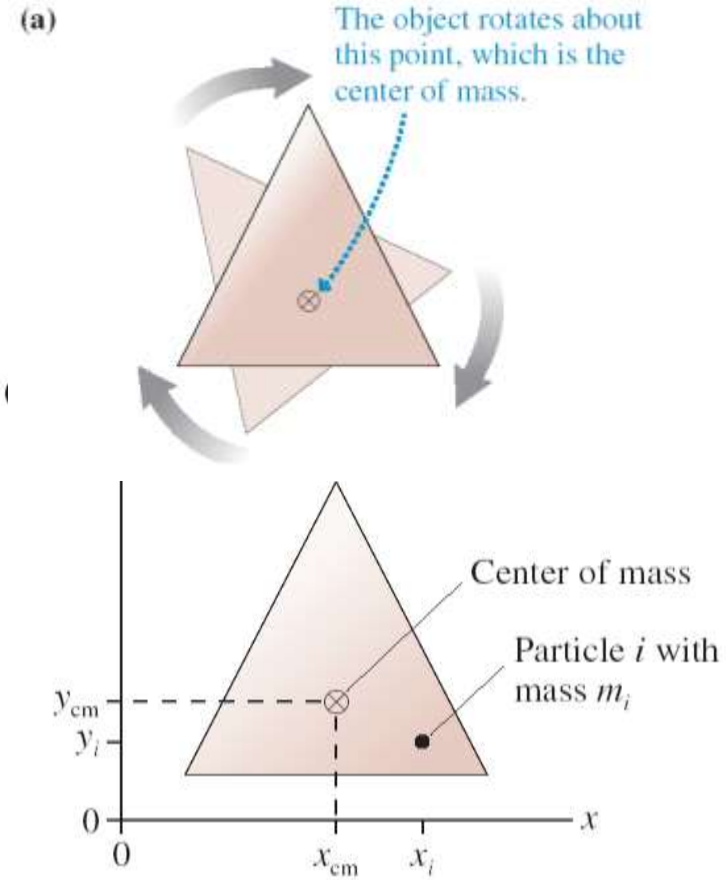


FIGURE 12.5 Rotation about the center of mass.



Moment of Inertia

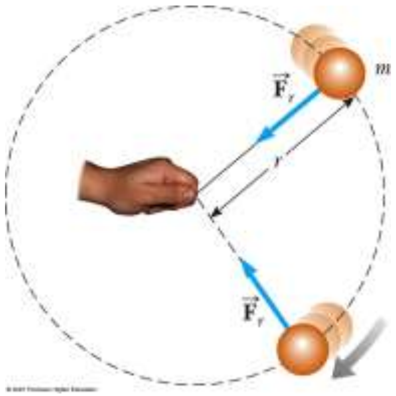
Definition

Measure of resistance of an object to changes in its rotational motion.
Equivalent to mass in linear motion.

For a single particle, the definition of moment of inertia is

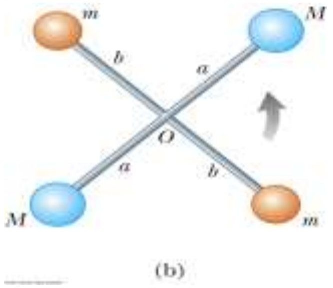
- *m* is the mass of the single particle
- *r* is the rotational radius

$$I = mr^2$$



For a composite particle, the definition of moment of inertia is

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + \ldots$$



- m_i is the mass of the i_{th} single particle
- r_i is the rotational radius of i_{th} particle
- SI units of moment of inertia are kg·m^2

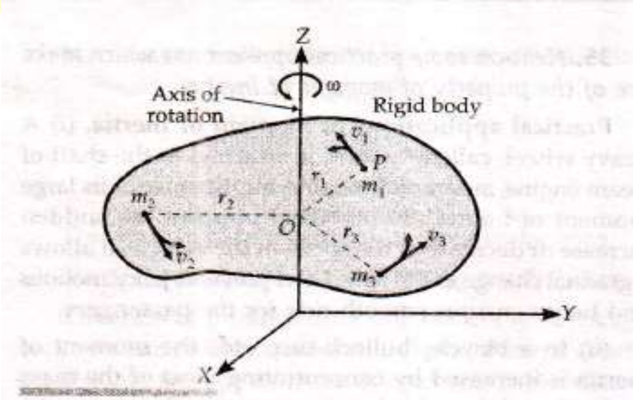


Fig. 7.36 M.I. and rotational K.E. of a rigid body.

Example 1

Three mass point m1,m2 and m3 are located at the vertices of an equilateral triangle a. What is the moment inertia of the system about an axis along the altitude of the triangle passing through m1.

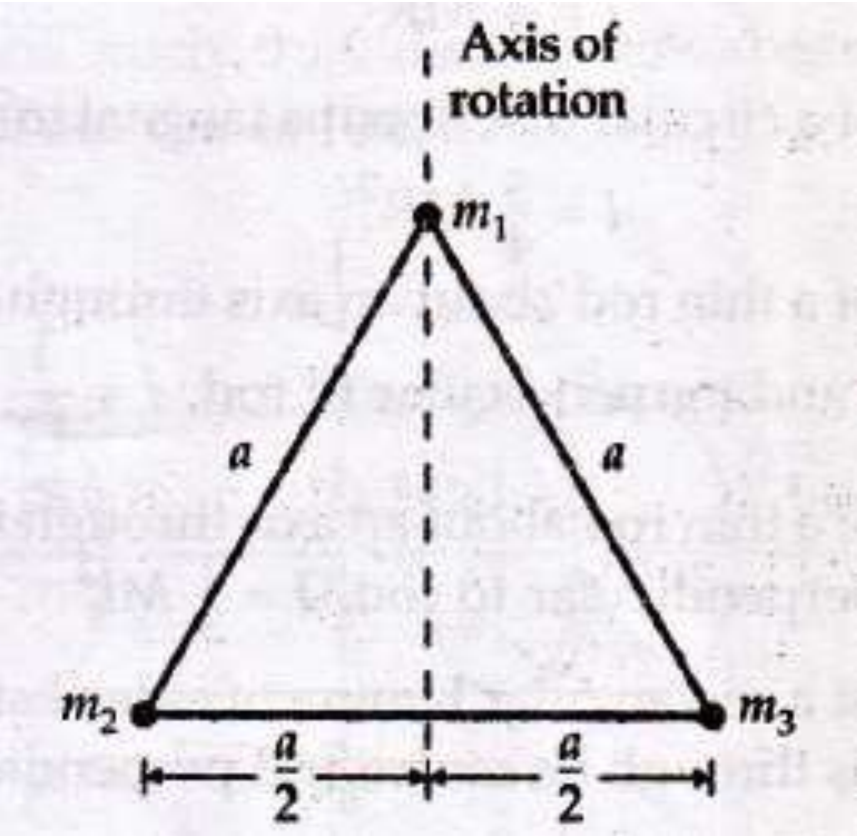
Solution. As shown in Fig. 7.54, the axis of rotation passes through m_1. The distances of m_1, m_2 and m_3 from the axis of rotation are 0, $a/2$ and $a/2$ respectively.

$\therefore$ M.I. of the system about the altitude through m_1 is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$= m_1 (0)^2 + m_2 \left(\frac{a}{2}\right)^2 + m_3 \left(\frac{a}{2}\right)^2$$

or $$I = \frac{a^2}{4}(m_2 + m_3).$$



Calculation of Moments of Inertia

It is a system made up of an infinity of identical material points, each having an elementary mass *dm*. These systems are distinguished according to the distribution of points in:

OR Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

The moment of inertia for the large rigid object is

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

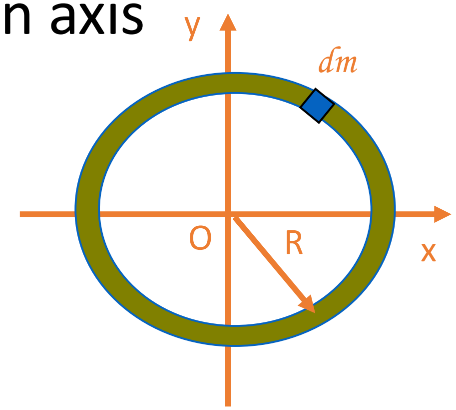
$$\rho = \frac{dm}{dV} \qquad dm = \rho dV$$

The moments of inertia becomes

$$I = \int \rho r^2 dV$$

Example 2:

Find the moment of inertia of a uniform hoop of mass ***M*** and radius ***R*** about an axis perpendicular to the plane of the hoop and passing through its center.



The moment of inertia is

$$I = \int r^2 dm = R^2 \int dm = MR^2$$

Moment of Inertia of a Uniform Rigid Rod

The shaded area has a mass

$dm = \lambda\, dx$

Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{12} ML^2$$

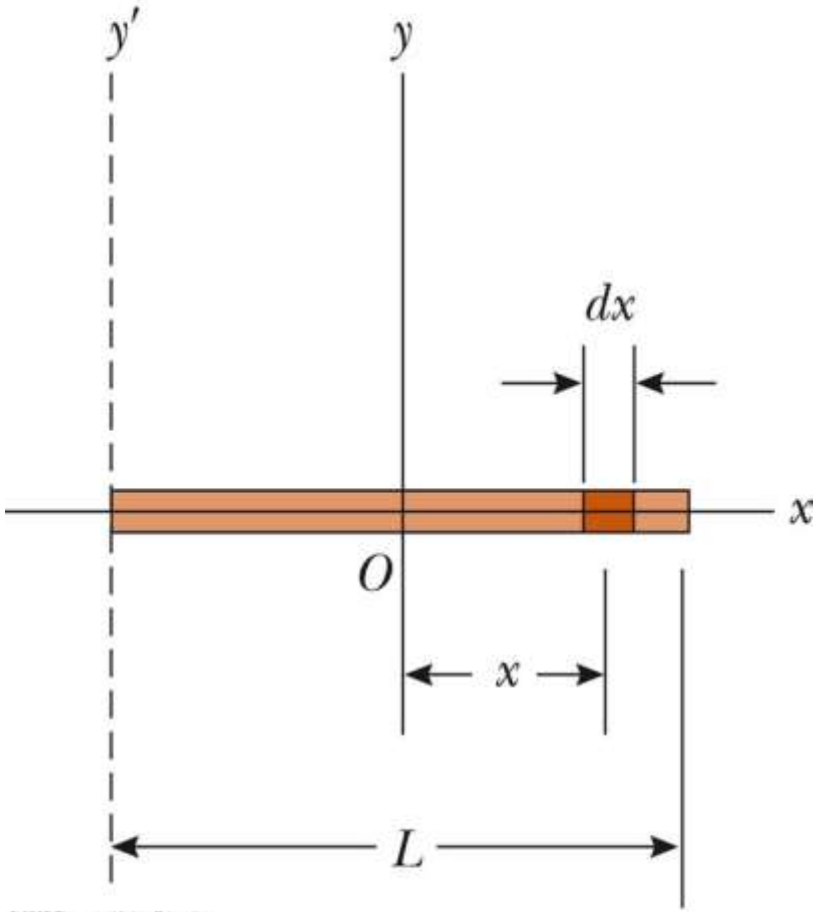
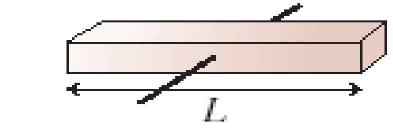
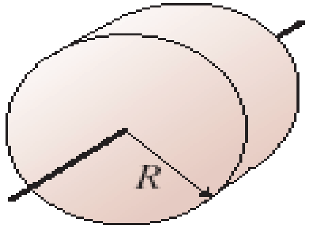
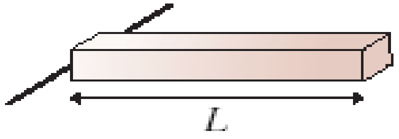
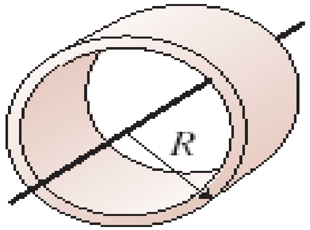
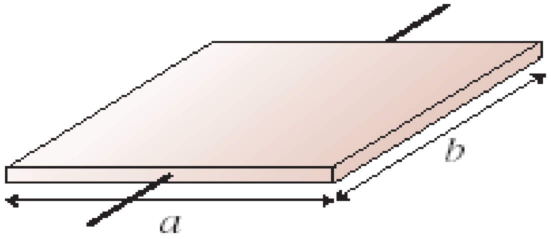
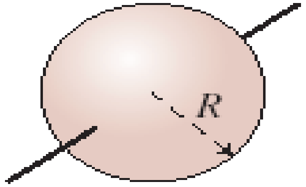
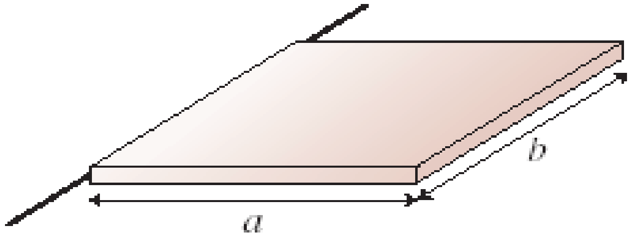
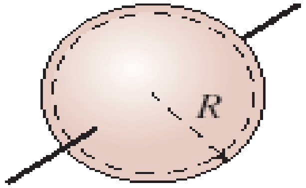


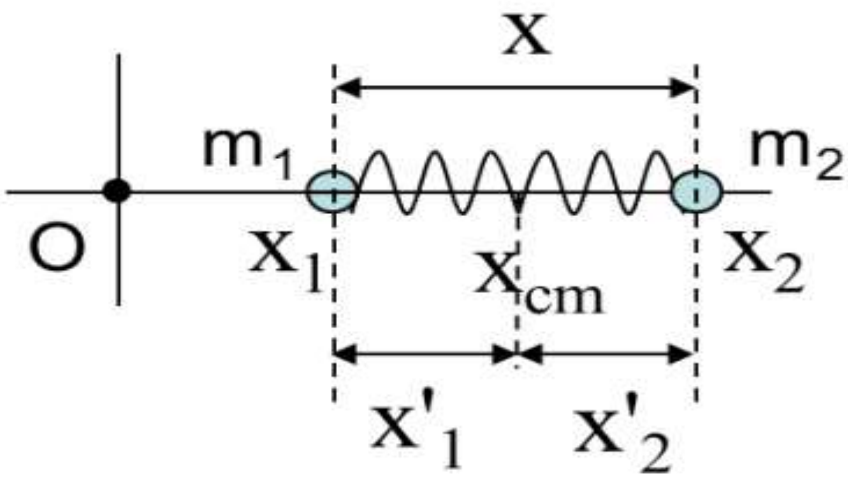
TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Example : SHM of two connected masses in 1D

SHM between two masses m_1 and m_2 connected by a spring

- $x = x_2 - x_1$; Natural length L
- $F_{int} = -k\,(x - L) = \mu\,\ddot{x}$
  ($\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass)
- $\ddot{x} + \frac{k}{\mu}(x - L) = 0$



Solution: $x = x_0 \cos(\omega t + \phi) + L$

where $\omega = \sqrt{\frac{k}{\mu}}$

With respect to the CM:

- $x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}$ where $M = m_1 + m_2$
- $x_1' = x_1 - x_{cm} = \frac{Mx_1 - m_1 x_1 - m_2 x_2}{M} = -\frac{m_2 x}{M}$
- $x_2' = x_2 - x_{cm} = \frac{Mx_2 - m_1 x_1 - m_2 x_2}{M} = \frac{m_1 x}{M}$

Eg. take $m_1 = m_2 = m \;\rightarrow\; \omega = \sqrt{\frac{2k}{m}}$; $x_1' = -\frac{1}{2}x$, $x_2' = \frac{1}{2}x$

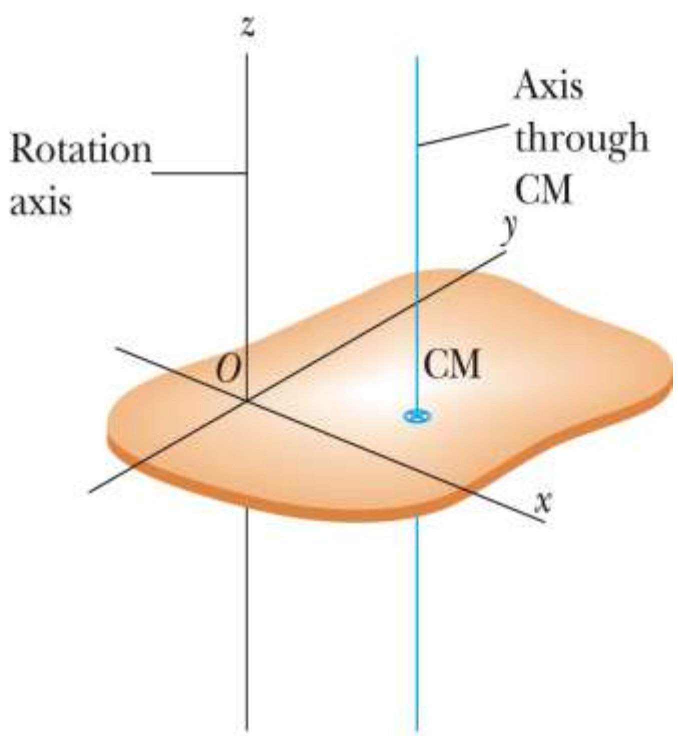
Parallel-Axis Theorem

For an arbitrary axis, the parallel-axis theorem often simplifies calculations

The theorem states

$$I = I_{CM} + MD^2$$

- I is about any axis parallel to the axis through the center of mass of the object
- I_{CM} is about the axis through the center of mass
- D is the distance from the center of mass axis to the arbitrary axis



(b)

Moment of Inertia of a Uniform Rigid Rod

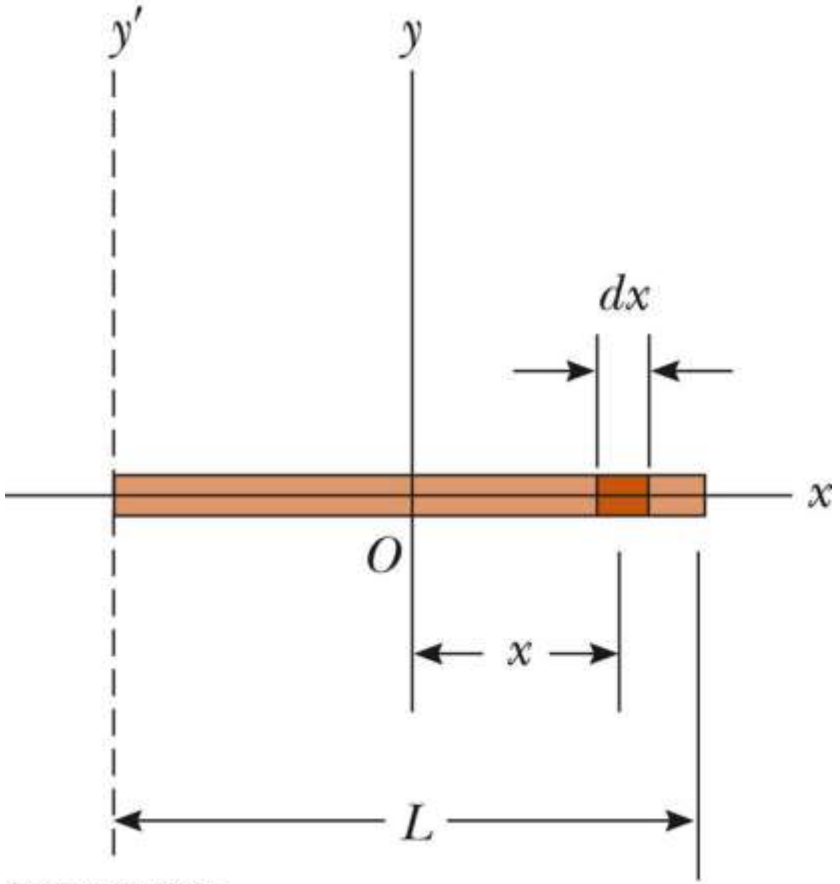
The moment of inertia about ***y*** is

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

The moment of inertia about ***y'*** is

$$I_{y'} = I_{CM} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$



Planar kinetics of rigid body: Force and acceleration

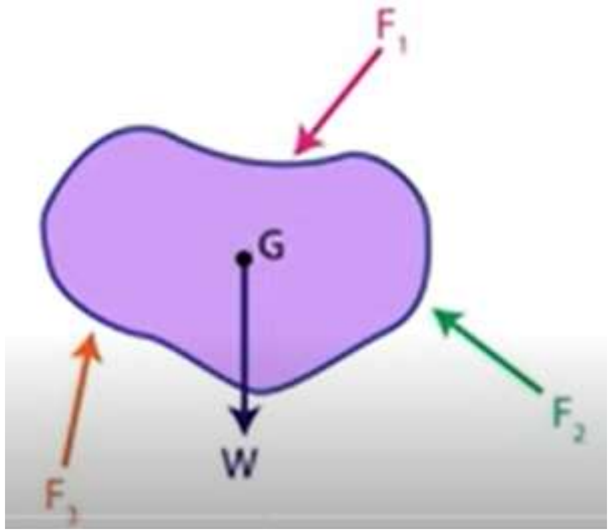
Planar Kinetic equations of motion: Equation of translational motion

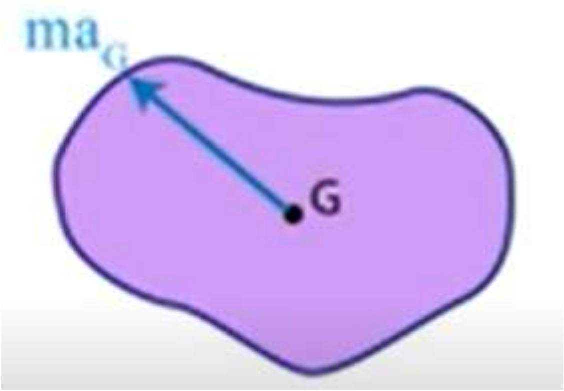
A-Rectilinear motion

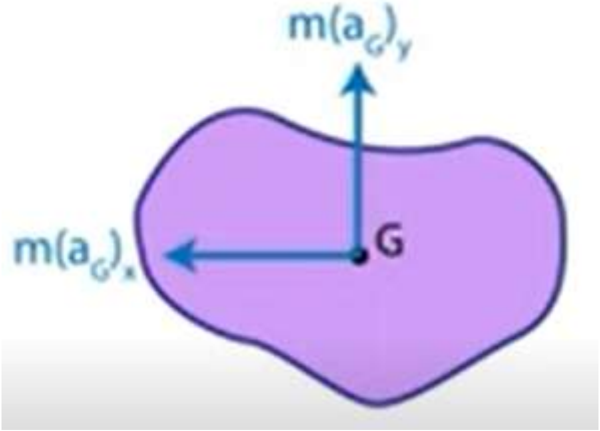
If a body undergoes translational motion, the equation of motion is $\Sigma \boldsymbol{F} = m\, \boldsymbol{a}_G$

The sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center.

The scalar form as: $\Sigma F_x = m(a_G)_x$ and $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = 0$







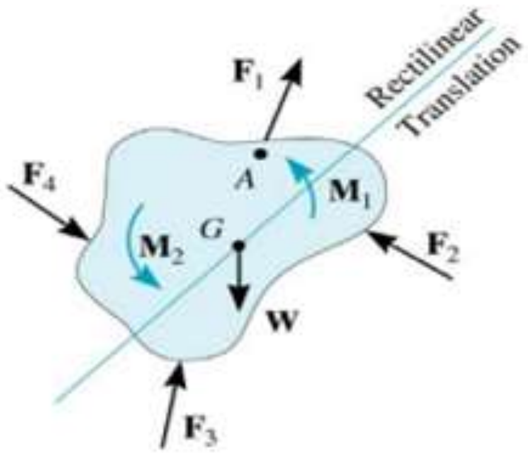
Equation of motion Translation

For ***a planar rectilinear translation***, all the particles of the body have the same acceleration so $a_G = a$ and $\alpha = 0$. The equation of motion are as follows.

$$\Sigma F_x = m(a_G)_x$$

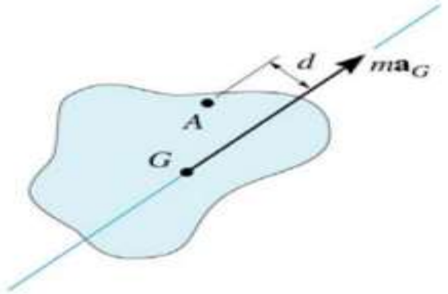
$$\Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = 0$$



The moment equation can be applied about other points A , instead of the mass center. In this case.

$$\Sigma M_A = r_G \times m a_G = (m\, a_G)\, d\,.$$



B-Curvilinear motion

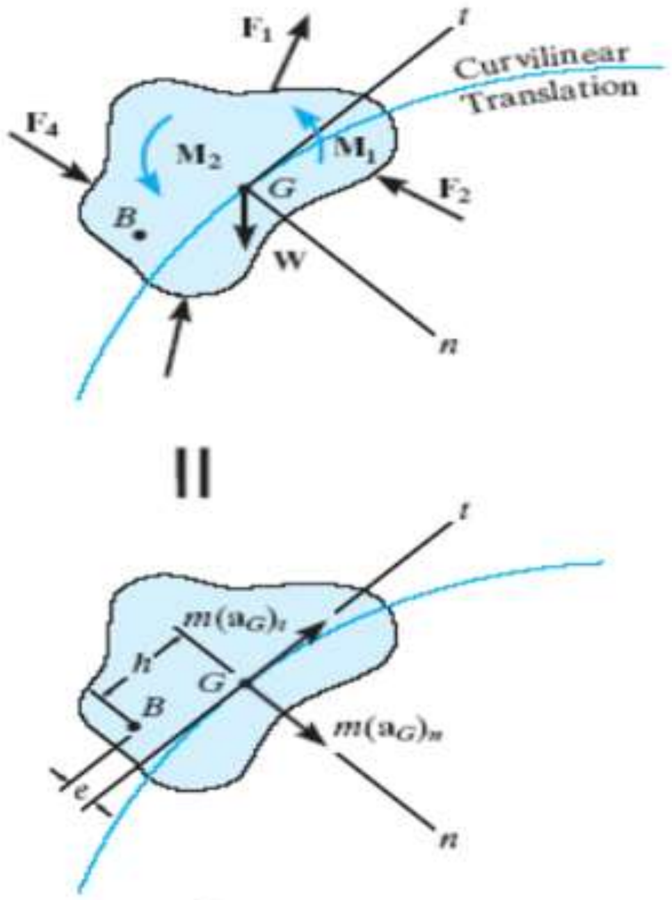
Similarly, for ***a planar curvilinear translation***, all the particles of the body have the same accelerati $\Sigma \boldsymbol{F} = m\,\boldsymbol{a}_G$ equation of motion are as follows, for n-t coordinates.

$$\Sigma F_n = m(a_G)_n \qquad \Sigma F_n = m\,(a_G)_n = m\,r_G\,\omega^2$$

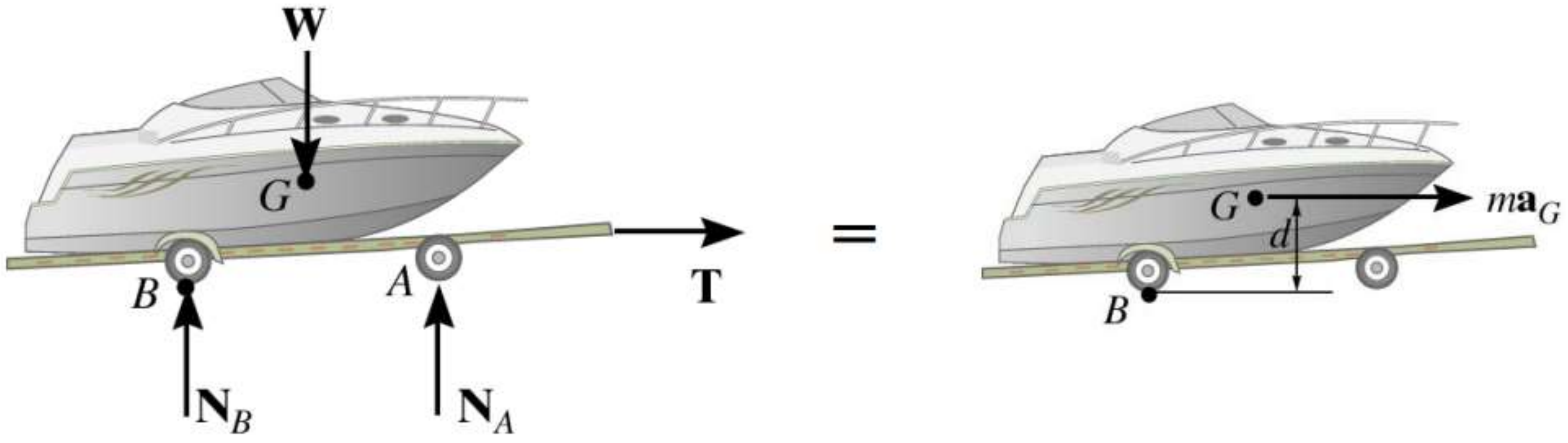
$$\Sigma F_t = m(a_G)_t \qquad \Sigma F_t = m\,(a_G)_t = m\,r_G\,\alpha$$

$$\Sigma M_G = 0 \quad \text{or}$$

$$\Sigma M_B = \boldsymbol{r}_G \times m\boldsymbol{a}_G = e[m(a_G)_t] - h[m(a_G)_n]$$



The boat and trailer undergo rectilinear motion. In order to find the reactions at the trailer wheels and the acceleration of the boat at its center of mass, we need to draw the FBD for the boat and trailer.

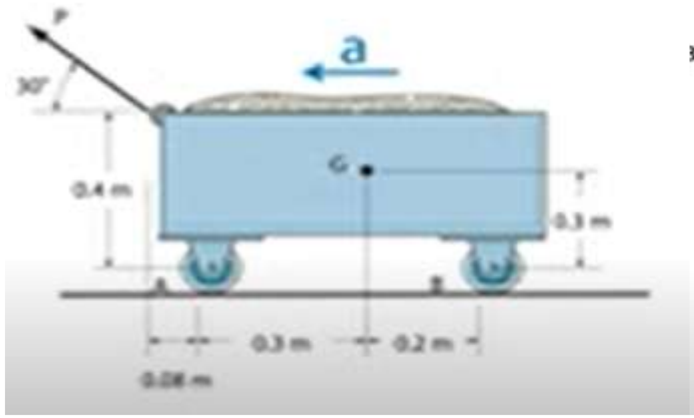


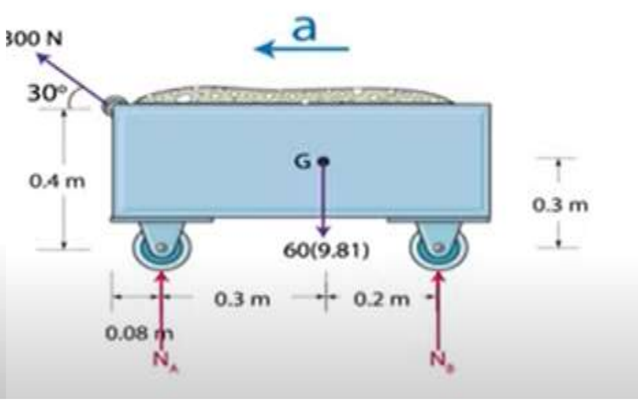
Free body diagram

Kenetic diagram

Example 3

A force of ***P= 300 N*** is applied to the ***60 kg*** cart. Determine **the reactions** at both the wheels at ***A*** and the wheels at ***B***. Also, what is the acceleration of the cart ? The mass center of the cart is at G.





$$\stackrel{+}{\leftarrow} \sum F_x = m(a_G)_x$$

$$300 \cos 30° = 60a$$

$$a = 4.33 \text{ m/s}^2$$

$$+\uparrow \sum F_y = m(a_G)_y$$

$$N_A + N_B + 300 \sin 30° - 60(9.81) = 60(0)$$

$$\circlearrowleft + \sum M_G = 0$$

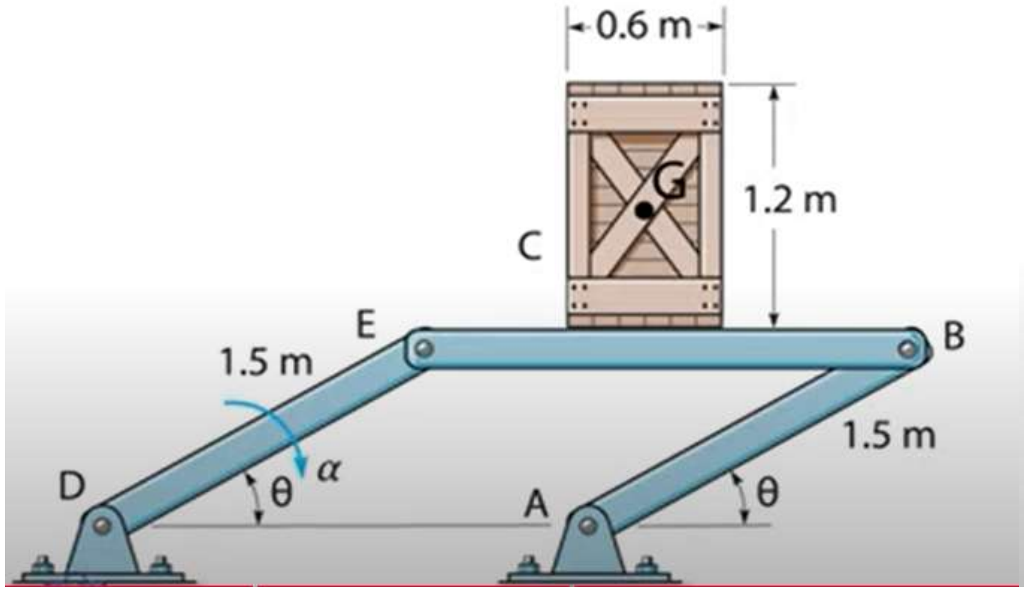
$$-N_A(0.3) + N_B(0.2) - 300 \sin 30° (0.38) + 300 \cos 30° (0.1) = 0$$

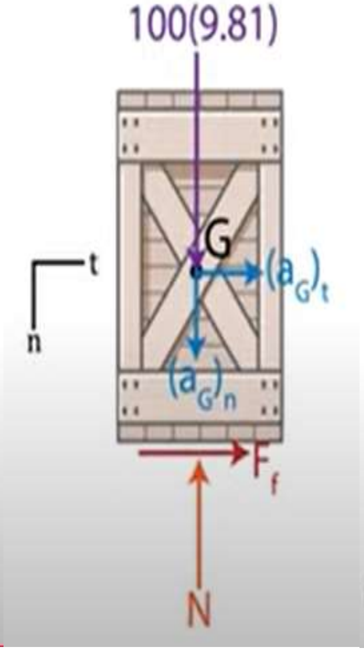
$$N_A = 113.4 \text{ N}$$

$$N_B = 325.2 \text{ N}$$

Example 4

The **100 kg** uniform crate ***C*** rests on the elevator floor where the coefficient of statics friction is ***μ_s= 0.4***. **Determine** the largest initial angular acceleration α, starting from rest at ***θ=90°***, without causing the crate to slip. No tipping occurs.





$$(a_G)_n = 0$$

$$\downarrow^+ \sum F_n = m(a_G)_n$$

$$100(9.81) - N = 100(0)$$

N=981 N

$$\omega = 0 \text{ rad/s}$$

$$(a_G)_n = \omega^2 r$$

$$(a_G)_n = 0$$

$$(a_G)_t = \alpha(1.5)$$

$$\rightarrow^+ \sum F_t = m(a_G)_t$$

$$(0.4)(981) = 100(1.5\alpha)$$

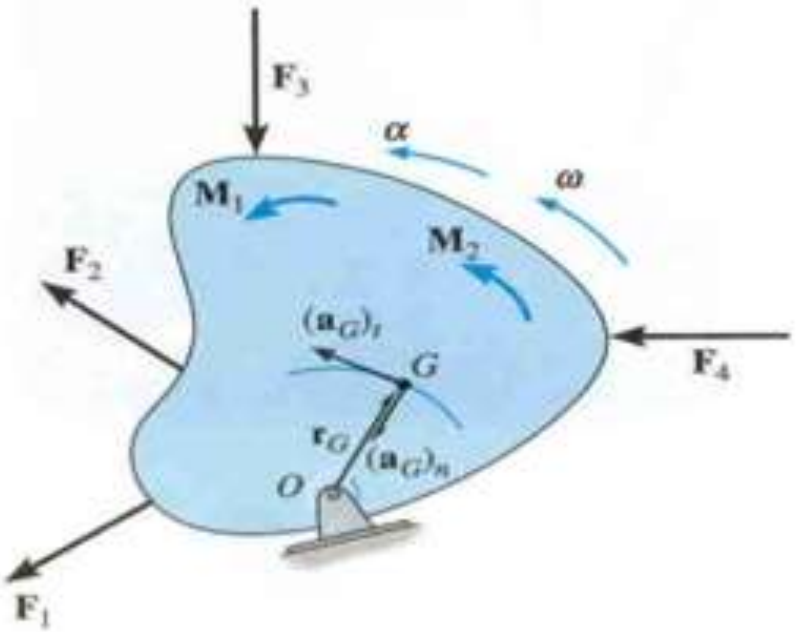
$$(a_G)_t = \alpha r$$

$$\alpha = 2.616 \text{ rad/s}^2$$

Equation of motion for pure rotation

When a rigid body rotates about a fixed axis perpendicular to the plane of the body at point a O, the body,s center of *gravity G* moves in a circular path of radius *r_G*. Thus, the acceleration of point G can be represented by *a tangential component* $(a_G)_t = r_G\,\alpha$ and *a nomal component* $(a_G)_n = r_G\,\omega^2$. The *scalar equations* of motion can be states as:

$$\sum F_n = m\,(a_G)_n = m\,r_G\,\omega^2$$
$$\sum F_t = m\,(a_G)_t = m\,r_G\,\alpha$$
$$\sum M_G = I_G\,\alpha$$



Equation rotational motion

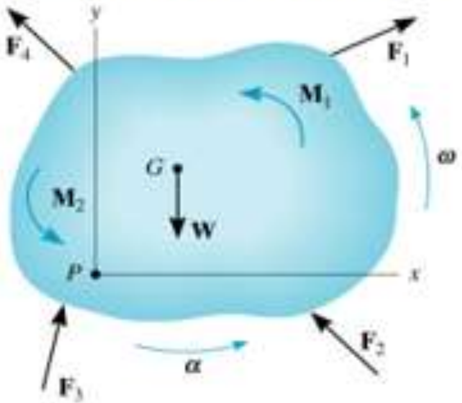
We need to determine the effects caused by the moments of the external force system. The moment about point P can be written as

$$\Sigma (\boldsymbol{r}_i \times \boldsymbol{F}_i) + \Sigma \boldsymbol{M}_i = \boldsymbol{r}_G \times m\boldsymbol{a}_G + I_G \boldsymbol{\alpha}$$

$$\Sigma M_p = \Sigma(M_k)_p$$

Wont have this equation if it's particle

where ΣM_p is the resultant moment about P due to all the external forces. The term $\Sigma(\mathbf{M}_k)_p$ is called the kinetic moment about point P.



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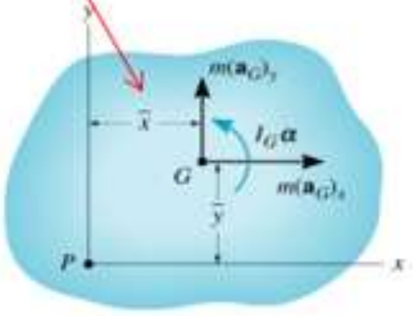

Kinetic diagram

Example 5

The uniform 24 kg plate is release from rest shown. **Determine** its initial angular acceleration and the horizontal acceleration and the horizontal and vertical reactions at the pin ***A***.