Linear programming (LP)

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I. The simplex algorithm

- The simplex algorithm searches for the optimal solution by moving from one feasible basis to another that improves the value of the objective function.
- **feasible basis**

  Corresponding to the extreme points of the convex polyhedron. These are highlighted in a particular algebraic representation of the PL, called **standardised form**.

II Standardised form of a LP

To obtain the standardised form of a LP, all inequalities other than the non-negativity inequality must be compensated by adding a non-negative auxiliary variable to each inequality.

- If the inequality is of the type (≤), i.e.: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i$,a non-negative variable called the slack variable is added to the left-hand side of the inequality.

- If the inequality is of the type (≥), i.e.: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$, a non-negative variable called the surplus variable is subtracted from the left-hand side of the inequality.

- If a constraint is expressed as an equality, i.e. $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$ then no change is required.

II.1 Simplex table

BV	x_1	x_2	…	x_n	x_{n+1}	…	x_{n+m}	b_i
x_{n+1}	a_{11}	a_{12}	…	a_{1n}	1	0	0	b_1
x_{n+2}	a_{21}	a_{22}	…	a_{2n}	0	1	0	b_2
…	…	…	…	…	0	…	0	….
x_{n+m}	a_{m1}	a_{m2}	…	a_{mn}	0	0	1	b_n
Z	c_1	c_2	…	c_n	0	0	0	0

II.2 General form

- The general formulation of the standard form is written as follows:

Max Z = $c_1x_1+c_2x_2+\ldots+c_nx_n$.

$$\begin{cases} a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n+x_{n+i}=b_i, b_i\geq 0, i=1..k \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n=b_i, b_i\geq 0, i=k+1..r \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n-x_{n+i}=b_i, b_i\geq 0, i=r+1..m \\ x_1, x_2, \ldots, x_n\geq 0 \end{cases}$$

Example : Write the following PL in the standardised form.

Maximise $Z= 4X_1+2X_2+3X_3$

Subject to constraints :

$$\begin{cases} x_1+x_2+x_3 \leq 12 \\ 3x_1+2x_2+7x_3 \leq 20 \\ 3x_1+x_2+4x_3 \leq 10 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

III. The simplex algorithm

- Initialization
  - Introduce the deviation variables and calculate an initial feasible dictionary: set variables outside the basis to 0.
- Simplex loop
  - **While** (there is a variable outside x_k with a positive coefficient in the expression of z) **do**
    - Let x_i=… be the most restrictive constraint for increasing x_K.
    - Rotate to remove x_i from the base by inserting x_K into it.
- Return the solution.

III. The Simplex Algorithm

- Consider the linear program:

Max z = $3x_1+4x_2$

$$\begin{cases} x_1+2x_2 \le 21 \\ x_1+x_2 \le 12 \\ x_1, x_2 \ge 0 \end{cases}$$

III. The Simplex Algorithm

- By introducing the deviation variables x_3 and x_4, we obtain the equivalent program:

Max $z = 3x_1 + 4x_2$

$$\begin{cases} x_1 + 2x_2 + x_3 = 21 \\ x_1 + x_2 + x_4 = 12 \\ x_1, x_2 \geq 0, x_3, x_4 \geq 0 \end{cases}$$

III.1 Execution of the simplex algorithm

- We have: $A=\begin{pmatrix}1 & 2 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix},\ b=\begin{pmatrix}21\\ 20\end{pmatrix},\ C=[3,4,0,0]$

  .

- We see that B = {3, 4} is a basis set. The corresponding basis solution is:

  $x_1 = 0$, $x_2 = 0$, $x_3 = 21$, $x_4 = 12$.

- The solution is called an admissible basic solution.

III.1 Execution of the simplex algorithm

- **The choice of the incoming variable.** So the incoming variable (in our case x_2) is chosen using the rule:
  - The coefficient in the Z function of the input variable is the largest of the (positive) coefficients of Z.
  - Hoping thus for a stronger increase in Z compared to other choices of input variables.

III.1 Execution of the simplex algorithm

- **Choosing the outgoing variable.** We will increase the value of x_2 as much as possible while remaining within the feasible set.
- We calculate the minimum ratio as follows:

  $\theta = \min(21/2, 12) = 21/2$. The output variable from the basis is x_3.
- The outgoing variable cannot appear in multiple equations, because it is (was) a basic variable. So, after pivoting, which consists of doing the following:

  $L_1 \leftarrow (1/2)L_1$ and then $L_2 \leftarrow L_2 - (1/2)L_1$, we obtain the system.

III.1 Execution of the simplex algorithm

$$\begin{cases} 1/2\,x_1 + x_2 + 1/2\,x_3 = 21/2 \\ 1/2\,x_1 - 1/2\,x_3 + x_4 = 3/2 \\ x_1, x_2 \geq 0, x_3, x_4 \geq 0 \end{cases}$$

- B'={2, 4} is indeed a basic set, and the associated basic solution is: $x_1 = 0$, $x_2 = 21/2$, $x_3 = 0$, $x_4 = 3/2$, which is admissible.

III.1 Execution of the simplex algorithm

- **The reduced objective.** We will always express the objective in terms of variables outside the basis. So in our case, after pivoting (which amounts to $L_z \leftarrow L_z - 2L_1$), we obtain : $x_1 - 2x_3 = Z - 42$.
- **The new program is:**

$x_1 - 2x_3 = Z - 42.$

$$\begin{cases} 1/2\,x_1 + x_2 + 1/2\,x_3 = 21/2 \\ 1/2\,x_1 - 1/2\,x_3 + x_4 = 3/2 \\ x_1, x_2 \geq 0, x_3, x_4 \geq 0 \end{cases}$$

III.1 Execution of the simplex algorithm

- We see that it is still possible to increase Z by increasing x_1. So we will choose x_1 as the input variable.
- To determine the outgoing variable, we calculate :
  min (21/2 : 1/2; 3/2 : 1/2) = 3, so it is the second equation, which corresponds to the basic variable x_4.
- Therefore, the pivot is 1/2, which is the coefficient of x_1 in the second equation.

III.1 Execution of the simplex algorithm

- After the transformations $L_2 \leftarrow 2L_2$, $L_1 \leftarrow L_1 - L_2$, $L_z \leftarrow L_z - 2L_2$ are performed, we obtain :

$$\begin{cases} x_2 + x_3 - x_4 = 9 \\ x_1 - x_3 + 2x_4 = 3 \\ -x_3 - 2x_4 = Z - 45 \end{cases}$$

III.1 Execution of the simplex algorithm

- We see that the new basic set is $\{x_2, x_1\}$, the corresponding basic solution is ($x_1 = 3$, $x_2 = 9$, $x_3 = 0$, $x_4 = 0$), and all the coefficients of Z are negatives, so the basic solution is optimal.
- So we have $Z_{max} = 45$ and $x_1^* = 3$, $x_2^* = 9$ is the optimal solution of the initial program (or we must add ($x_3^* = 0$, $x_4^* = 0$) for the program transformed into standard form)

- **Exercise**. Solve the same problem using the graphical method and compare with the steps taken in the simplex method.

IV. The table method

Initial table

BV	x_1	x_2	x_3	x_4	b_i
x_3	1	2	1	0	21
x_4	1	1	0	1	12
Z	3	4	0	0	0

First iteration

BV	x_1	x_2	x_3	x_4	b_i
x_3	1/2	1	1/2	0	21/2
x_4	1/2	0	-1/2	1	3/2
Z	1	0	-2	0	-42

IV. The table method

Second iteration

BV	x_1	x_2	x_3	x_4	b_i
x_3	0	1	1	-1	9
x_4	1	0	-1	2	3
Z	0	0	-1	-2	-45

The optimal solution is:
$X_1^*=3$, $X_2^*=9$, $X_3^*=X_4^*=0$.
$Z^*=-Z=45$.

V. General case

- Consider a standard program and assume that a basic set B = {1,...,m} is known to be admissible. Let N = {1,...,n} denote the out-of-base set.
- We can therefore assume that the program is given in reduced form associated with the basic set B:

V. General case

- Max Z = CX

  $X_B + A_B^{-1} A_N X_N = A_B^{-1} b$

  $X_B \geq 0,\ X_N \geq 0$

  with:

$$Z = \sum_{i=1}^{n} c_i x_i = \sum_{i \in B} c_i x_i + \sum_{J \in N} c_j x_j = C_B X_B + C_N X_N$$

V. General case

- Let's pose. $\tilde{A} = A_B^{-1}A_N$ and $\tilde{b} = A_B^{-1}B$.
- Therefore we have: $X_B = -\tilde{A}_N X_N + \tilde{b}$

  $Z = (C_N - C_B\tilde{A}_N)X_N + C_B\tilde{b} = d_N X_N + Z_B$
- Putting : $d_N = C_N - C_B\tilde{A}_N$ et $Z_B = C_B\tilde{b}$. The problem then becomes:

  $X_B + \tilde{A}_N X_N = \tilde{b}$

  $D_N X_N = Z - Z_B$

V. General case

- Note that the basic solution is given by:

  $X_N = 0$, $X_B = \tilde{b}$ and the value of the objective Z associated with this solution is Z_B.

V.1 The simplex table

The simplex table associated with base B is of the form:

BV	**x_N**	**x_B**	**SM**
X_B	A_N	I_M	$\tilde{b}$
Z	D_N	0	Z_B

V.1 The simplex table

We can see that the third column of the table is unnecessary!
So we just need to keep the table:

BV	$\mathbf{x_N}$	**SM**
$\mathbf{x_B}$	A_N	$\tilde{b}$
Z	D_N	Z_B

However, for the convenience of manual calculation, we will use the complete table.

IV. Simplex theorem

- Remember that once the pivot has been established, the transition from a simplex table to the next table is carried out using the following rules:
  - I) We divide the pivot line by the pivot.
  - II) All other elements of the new table (including the second member or the z row) are obtained using the rectangle rule (calculations can be simplified by knowing that the columns of the basic variables correspond to the columns of the unit matrix).

IV. Simplex theorem

- **Theorem**

  Suppose that the linear program is non-degenerate and has at least one optimal solution (with finite Z_{max}). Then, at each iteration of the simplex, the choice of the pivot row (i.e. the outgoing variable) is unique and the objective function increases strictly. Since the number of basic sets is finite (it is less than or equal to c^n_m, it follows that after a finite number of iterations, the algorithm will find an optimal basic solution.

Thank you !