Introduction to Integrals

Analysis 1

Objectives

- Understand the concept of indefinite and definite integrals
- Connect derivatives with integrals (anti-derivatives)
- Interpret the definite integral geometrically
- Recognize conditions for existence of integrals

Indefinite Integral

If

$$\frac{dy}{dx} = f(x)$$

then y is called an **indefinite integral** or **anti-derivative** of $f(x)$.

Notation:

$$y = \int f(x)\, dx$$

The Constant of Integration

Important fact

Derivative of a constant is zero.

Therefore, if $F(x)$ is an anti-derivative of $f(x)$, then

$$F(x) + C$$

is also an anti-derivative for any constant C.

All indefinite integrals of $f(x)$ differ only by a constant.

Definite Integral as a Limit of a Sum

The **definite integral** of $f(x)$ from $x = a$ to $x = b$ is defined as:

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\big[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)\big]$$

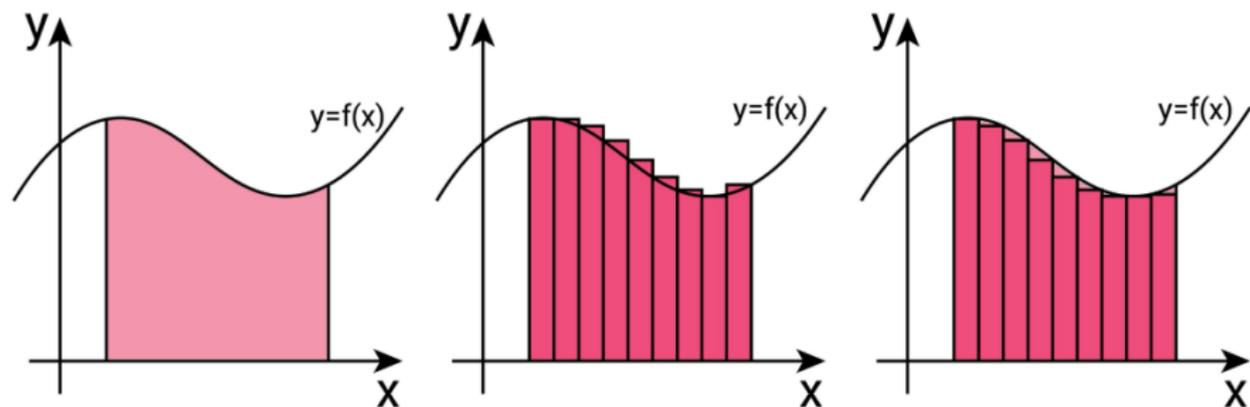where $h = \frac{b-a}{n}$ and $n \to \infty$.

Geometric Interpretation

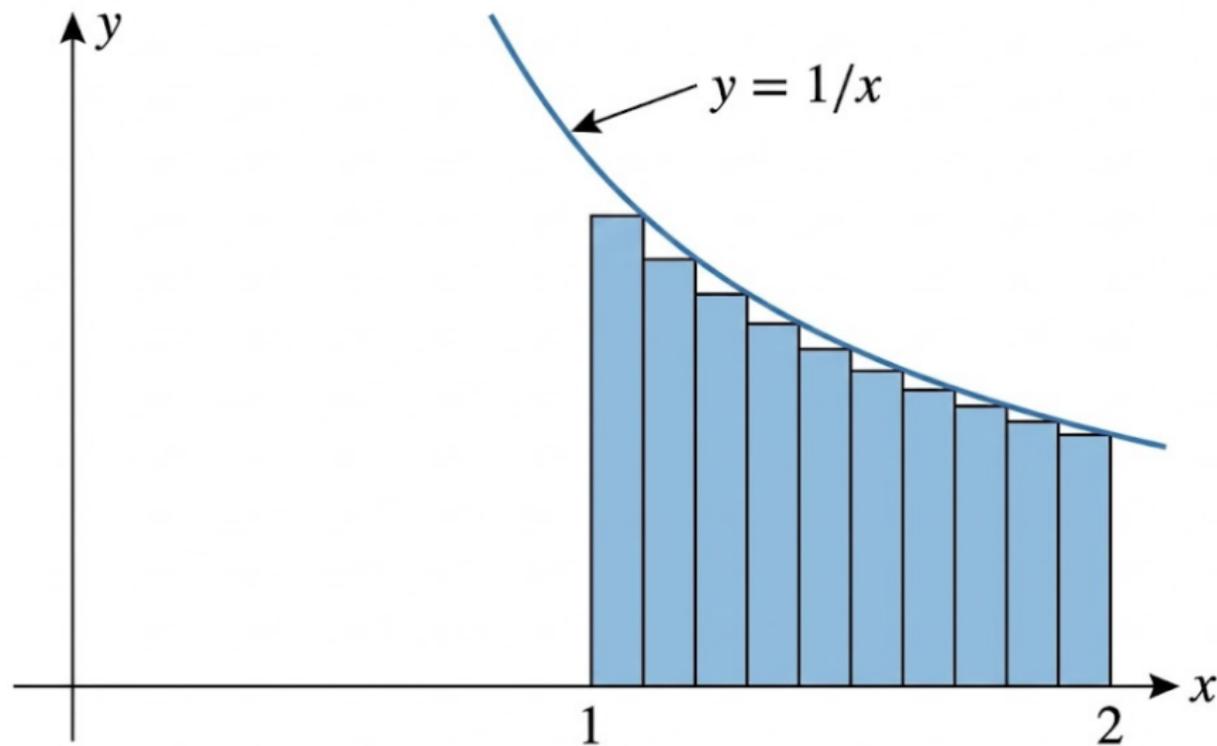If $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)\, dx$$

represents the **area under the curve** $y = f(x)$ from $x = a$ to $x = b$.
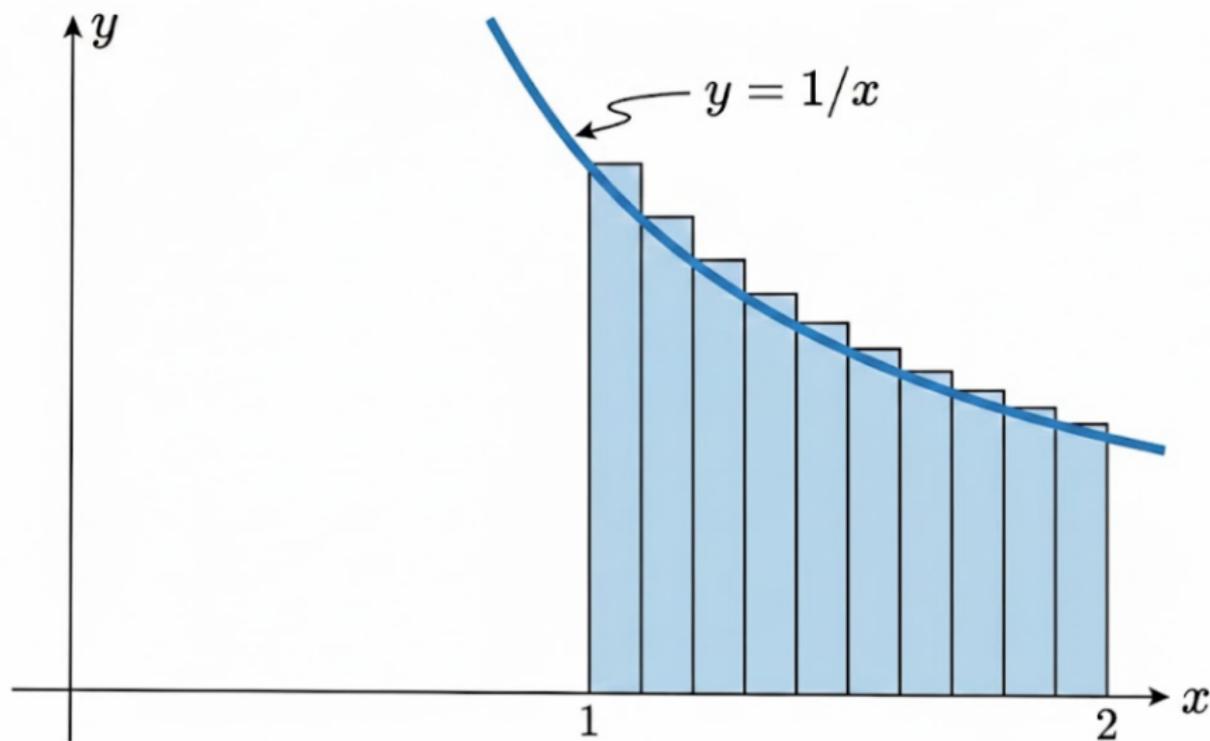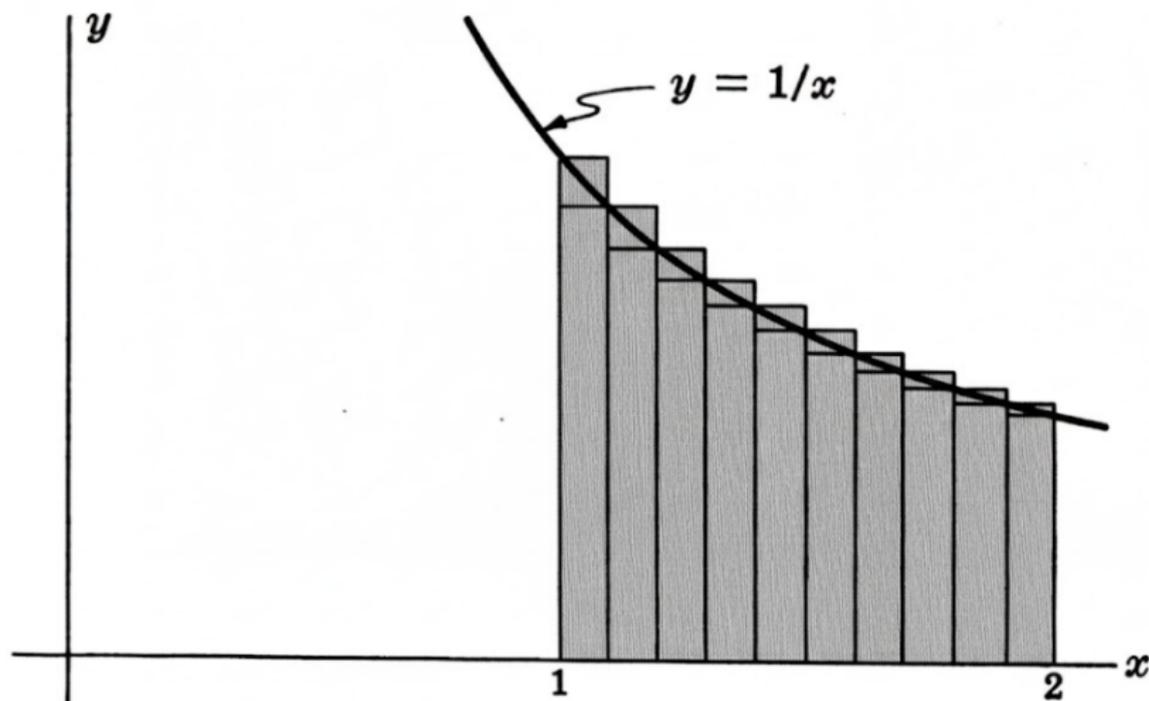
Geometric Interpretation

Geometric Interpretation (Exercise 1 in tutorial)

Geometric Interpretation (Exercise 1 in tutorial)

Geometric Interpretation (Exercise 1 in tutorial)

When Does the Integral Exist?

Theorem (Existence)

If f is **continuous** on the closed interval $[a, b]$, then

$$\int_a^b f(x)\, dx$$

exists.

Connecting Definite and Indefinite Integrals

Fundamental Theorem of Calculus

If $F(x)$ is an anti-derivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This theorem bridges the two concepts:

- Indefinite integral (anti-derivative)
- Definite integral (limit of a sum / area)

Summary

- Indefinite integral: $\int f(x)dx = F(x) + C$
- Definite integral: $\int_a^b f(x)dx = \lim_{h \to 0} \sum f(a + kh) \cdot h$
- Geometric meaning: area under $f(x)$ from a to b (if $f(x) \geq 0$)
- Existence: guaranteed if f is continuous on $[a, b]$
- Fundamental Theorem links definite and indefinite integrals

Integration Formulas

Notation

- u, v represent functions of x
- a, b, c, p represent constants
- Constant of integration $+C$ is implied in all formulas

1. Linearity Property

Theorem (Sum/Difference Rule)

$$\int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx$$

Example

$$\int (x^2 + \sin x)\, dx = \int x^2\, dx + \int \sin x\, dx$$

Intuition

The integral of a sum/difference equals the sum/difference of the integrals.

2. Constant Multiple Rule

Theorem (Constant Factor Rule)

$$\int c \cdot u \, dx = c \int u \, dx$$

where c is a constant.

Example

$$\int 5x^3 \, dx = 5 \int x^3 \, dx$$

Combined with Rule 1

$$\int (au + bv) \, dx = a \int u \, dx + b \int v \, dx$$

This shows integration is a **linear operation**.

3. Integration by Parts

Theorem (Product Rule in Reverse)

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

Compact Notation (More Common)

$$\int u \, dv = uv - \int v \, du$$

Example

To integrate $\int x e^x \, dx$:

- Let $u = x$, $dv = e^x \, dx$
- Then $du = dx$, $v = e^x$
- Apply: $\int x e^x \, dx = x e^x - \int e^x \, dx$

Why Does Integration by Parts Work?

Derivation from Product Rule

Recall: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Integrate both sides:

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

Rearrange:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

When to Use

- Product of different function types (polynomial $\times$ exponential)
- Product of polynomial $\times$ trigonometric function
- Integrals involving logarithms

4. Integration by Substitution

Theorem (Change of Variable)

$$\int F[u(x)]\,dx = \int F(w)\frac{dw}{w'}$$

where:

- $w = u(x)$ (the substitution)
- $w' = \frac{dw}{dx}$ (the derivative)
- The right side is expressed in terms of w

Common Formulation

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Substitution Method: Step-by-Step

1. **Choose substitution**: Let $u = g(x)$
2. **Compute differential**: $du = g'(x)dx$
3. **Substitute**: Replace $g(x)$ with u and dx with $\frac{du}{g'(x)}$
4. **Integrate**: $\int f(u)du$
5. **Back-substitute**: Replace u with $g(x)$

Example

Find $\int 2x \cos(x^2)dx$
1. Let $u = x^2$
2. $du = 2xdx$
3. $\int \cos(u)du$
4. $= \sin(u) + C$
5. $= \sin(x^2) + C$

Visualizing Substitution

Intuition

Substitution changes the variable to simplify the integrand, like changing coordinates in geometry.

Integration Formulas Summary

Formula	Name	When to Use
$\int (u \pm v)dx = \int u\,dx \pm \int v\,dx$	Linearity	Sums/differences
$\int cu\,dx = c \int u\,dx$	Constant Multiple	Constants multiplied
$\int u\,dv = uv - \int v\,du$	By Parts	Products of functions
$\int f(u(x))u'(x)dx = \int f(u)du$	Substitution	Composite functions

Remember

- These are the **building blocks** for more complex integrals
- Often used in combination
- Practice is essential to recognize which formula to apply

Practice Problem

Try This

Use the appropriate integration formula(s) to find:

$$\int (3x^2 + 2\sin x)\, dx$$

Practice Problem

Try This

Use the appropriate integration formula(s) to find:

$$\int (3x^2 + 2\sin x)\, dx$$

Solution

Using linearity (Rule 1) and constant multiple (Rule 2):

$$\int (3x^2 + 2\sin x)\, dx = 3\int x^2\, dx + 2\int \sin x\, dx$$

$$= 3\left(\frac{x^3}{3}\right) + 2(-\cos x) + C$$

$$= x^3 - 2\cos x + C$$

Looking Ahead

- These four formulas form the foundation of integration techniques
- More advanced methods build on these:
    - Trigonometric substitution
    - Partial fractions
    - Integration of rational functions
- Master these basic formulas first!

Key Takeaway

Integration is the inverse operation of differentiation, but requires recognizing patterns and selecting the right technique.

Integration Formulas Reference

Notation

- u represents a function of x
- a, b, p represent constants
- Constant of integration $+C$ is implied in all formulas

Note

These formulas assume the integrand is expressed in terms of du, meaning you may need to use substitution first.

Basic Power and Exponential Rules

Formula	Conditions
$\int w^p du = \dfrac{w^{p+1}}{p+1}$	$p \neq -1$
$\int u^{-1} du = \int \dfrac{du}{u} = \ln\lvert u \rvert$	
$\int a^u du = \dfrac{a^u}{\ln a}$	$a > 0, a \neq 1$
$\int e^u du = e^u$	

Example

- $\int x^3 dx = \frac{x^4}{4} + C$
- $\int \frac{dx}{x} = \ln\lvert x \rvert + C$
- $\int 2^x dx = \frac{2^x}{\ln 2} + C$
- $\int e^{3x} dx = \frac{1}{3} e^{3x} + C$ (after substitution)

Trigonometric Integrals (Part 1)

Integral	Result
$\int \sin u \, du$	$-\cos u$
$\int \cos u \, du$	$\sin u$
$\int \tan u \, du$	$-\ln |\cos u|$
$\int \cot u \, du$	$\ln |\sin u|$

Memory Aid

- The integral of sin is $-\cos$ (mind the minus sign!)
- The integral of cos is sin
- For tan and cot, think: "opposite of derivative"

Trigonometric Integrals (Part 2)

Integral	Result
$\int \sec u \, du$	$\ln \lvert \sec u + \tan u \rvert$
$\int \csc u \, du$	$\ln \lvert \csc u - \cot u \rvert$

Example

- $\int \sec^2 x \, dx = \tan x + C$ (not in table, but important!)
- $\int \csc^2 x \, dx = -\cot x + C$ (not in table, but important!)

Note

The secant and cosecant integrals are less common but useful for trigonometric substitution problems.

Exponential-Trigonometric Combinations

Integral	Result
$\int e^{au} \sin(bu)\, du$	$\dfrac{e^{au}(a\sin(bu) - b\cos(bu))}{a^2 + b^2}$
$\int e^{au} \cos(bu)\, du$	$\dfrac{e^{au}(a\cos(bu) + b\sin(bu))}{a^2 + b^2}$

Derivation Method

These formulas come from applying integration by parts **twice** and solving for the original integral.

Example

For $\int e^{2x} \sin(3x)dx$:

- $a = 2$, $b = 3$
- Result: $\dfrac{e^{2x}(2\sin(3x) - 3\cos(3x))}{13} + C$

Inverse Trigonometric and Algebraic Forms

Integral	Result
$\int \dfrac{du}{\sqrt{a^2 - u^2}}$	$\sin^{-1}\left(\dfrac{u}{a}\right)$
$\int \dfrac{du}{u^2 + a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right)$
$\int \dfrac{du}{\sqrt{u^2 - a^2}}$	$\ln\left\|u + \sqrt{u^2 - a^2}\right\|$
$\int \dfrac{du}{\sqrt{u^2 + a^2}}$	$\ln\left\|u + \sqrt{u^2 + a^2}\right\|$

Geometric Interpretation

These forms arise from trigonometric substitution:

- $\sqrt{a^2 - u^2}$ suggests $u = a\sin\theta$
- $\sqrt{u^2 + a^2}$ suggests $u = a\tan\theta$
- $\sqrt{u^2 - a^2}$ suggests $u = a\sec\theta$

Hyperbolic Integrals

Integral	Result
$\int \sinh u \, du$	$\cosh u$
$\int \cosh u \, du$	$\sinh u$

Relationship with Exponential

Remember definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Example

- $\int \sinh(3x)dx = \frac{1}{3}\cosh(3x) + C$
- $\int \cosh(2x)dx = \frac{1}{2}\sinh(2x) + C$

Complete Integration Formulas Reference

Type	Formula
Power	$\int w^p \, du = \frac{w^{p+1}}{p+1}$
Reciprocal	$\int \frac{du}{u} = \ln|u|$
Exponential	$\int a^u \, du = \frac{a^u}{\ln a}$
Exponential (e)	$\int e^u \, du = e^u$
Sine	$\int \sin u \, du = -\cos u$
Cosine	$\int \cos u \, du = \sin u$
Tangent	$\int \tan u \, du = -\ln|\cos u|$
Cotangent	$\int \cot u \, du = \ln|\sin u|$

Type	Formula
Secant	$\int \sec u \, du = \ln|\sec u + \tan u|$
Cosecant	$\int \csc u \, du = \ln|\csc u - \cot u|$
$e^{au}\sin(bu)$	$\frac{e^{au}(a\sin(bu) - b\cos(bu))}{a^2 + b^2}$
$e^{au}\cos(bu)$	$\frac{e^{au}(a\cos(bu) + b\sin(bu))}{a^2 + b^2}$
Inverse sine	$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right)$
Inverse tangent	$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$
Hyperbolic sine	$\int \sinh u \, du = \cosh u$
Hyperbolic cosine	$\int \cosh u \, du = \sinh u$

Using These Formulas

Remember: Most integrals require manipulation (substitution, algebraic simplification) before matching these forms.

Memory Tips and Strategies

Common Patterns

- **Derivative pairs**: $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x$
- **Sign changes**: $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x \, dx = -\cos x$
- **Logarithmic forms**: $\frac{d}{dx}(\ln|f(x)|) = \frac{f'(x)}{f(x)}$
- **Inverse trig**: Related to derivatives of $\sin^{-1}$, $\tan^{-1}$

Important

- Practice recognition
- Learn derivations
- Use substitution
- Check by differentiating

Study Strategy

Group similar formulas together (trigonometric, exponential, inverse trig) and learn the patterns rather than memorizing each one individually.

Practice Application

Challenge Problem

Find: $\int \frac{3x^2}{x^3+1} dx$

Practice Application

Challenge Problem

Find: $\int \frac{3x^2}{x^3+1}\,dx$

Solution Strategy

1. Notice the derivative of $x^3 + 1$ is $3x^2$
2. Let $u = x^3 + 1$, then $du = 3x^2\,dx$
3. The integral becomes $\int \frac{du}{u}$
4. Apply formula 6: $\int \frac{du}{u} = \ln|u| + C$
5. Back substitute: $\ln|x^3 + 1| + C$

Key Insight

The table gives $\int \frac{du}{u} = \ln|u|$, but you need to recognize when the integrand matches this form after substitution.

Summary

- The integration formulas table provides **templates** for common integrals
- Most real problems require **algebraic manipulation** or **substitution** first
- Group formulas by type for easier memorization
- Always check your answer by **differentiating**
- Practice is essential for recognizing which formula applies