Analysis I: Solutions of Tutorial Exercise Sheet 4

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Exercise 1: Comparison

For each function $f(x)$ as $x \to 0$, we examine the order relative to x by studying the limit

$$L = \lim_{x \to 0} \frac{f(x)}{x},$$

when it exists, or by bounding the ratio. According to the definitions:

- If L is a finite nonzero number, then $f(x)$ is of the **same order** as x.

- If $L = 0$, then $f(x)$ is of **higher order** than x.

- If $L = \infty$ (or $|f(x)/x| \to \infty$), then $f(x)$ is of **lower order** than x.

We now treat each case.

1. $f(x) = x^2$.

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0.$$

Hence, x^2 is of higher order than x. In Landau notation:

$$x^2 = o(x) \quad (x \to 0).$$

2. $f(x) = \sqrt{x^2 + x^4}$. Write $\sqrt{x^2 + x^4} = |x|\sqrt{1 + x^2}$. Then

$$\frac{f(x)}{x} = \frac{|x|}{x}\sqrt{1 + x^2}.$$

The two-sided limit does not exist because $|x|/x$ is $+1$ for $x > 0$ and -1 for $x < 0$. However,

$$\lim_{x \to 0} \left| \frac{f(x)}{x} \right| = \lim_{x \to 0} \sqrt{1 + x^2} = 1,$$

and for sufficiently small x we have

$$|x| \le |f(x)| \le \sqrt{2}\,|x|.$$

Therefore, $f(x)$ is of the same order as x. In Landau notation:

$$\sqrt{x^2 + x^4} = O(x) \quad (x \to 0).$$

3. $f(x) = \sin(3x)$.

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3 \ne 0.$$

Thus, $\sin(3x)$ is of the same order as x. In Landau notation:

$$\sin(3x) = O(x) \quad (x \to 0).$$

4. $f(x) = 2x \cos(x^3) \tan^2(x)$. As $x \to 0$, we have $\cos(x^3) \to 1$ and $\tan(x) \sim x$, so $\tan^2(x) \sim x^2$. Consequently,

$$f(x) \sim 2x \cdot 1 \cdot x^2 = 2x^3.$$

Then

$$\lim_{x \to 0} \frac{f(x)}{x} = 0.$$

Hence, $f(x)$ is of higher order than x.

Or with simple method, we have

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} 2\cos(x^3)\tan^2(x) = 2 \cdot 1 \cdot 0 = 0.$$

In Landau notation:

$$2x \cos(x^3) \tan^2(x) = o(x) \quad (x \to 0).$$

5. $f(x) = e^{2x}$. As $x \to 0$, $e^{2x} \to 1$ while $x \to 0$. Therefore,

$$\lim_{x \to 0} \frac{e^{2x}}{x} = \infty.$$

So e^{2x} is of lower order than x. Since it tends to a nonzero constant, it is of order zero. In Landau notation:

$$x = o(e^{2x}) \quad (x \to 0).$$

Exercise 2: Polynomial Expansion

1. Expand the polynomial $P(x) = x^4 - 5x^3 + 5x^2 + x + 2$ in powers of $x - 2$.

    We compute the Taylor expansion around $a = 2$:

    $$P(x) = \sum_{k=0}^{4} \frac{P^{(k)}(2)}{k!}(x - 2)^k.$$

    The derivatives are:

    $$\begin{aligned}
    P(x) &= x^4 - 5x^3 + 5x^2 + x + 2, \\
    P'(x) &= 4x^3 - 15x^2 + 10x + 1, \\
    P''(x) &= 12x^2 - 30x + 10, \\
    P'''(x) &= 24x - 30, \\
    P^{(4)}(x) &= 24.
    \end{aligned}$$

    Evaluating at $x = 2$:

    $$\begin{aligned}
    P(2) &= 0, \\
    P'(2) &= -7, \\
    P''(2) &= -2, \\
    P'''(2) &= 18, \\
    P^{(4)}(2) &= 24.
    \end{aligned}$$

    Hence,

    $$\begin{aligned}
    P(x) &= \frac{0}{0!} + \frac{-7}{1!}(x - 2) + \frac{-2}{2!}(x - 2)^2 + \frac{18}{3!}(x - 2)^3 + \frac{24}{4!}(x - 2)^4 \\
    &= -7(x - 2) - (x - 2)^2 + 3(x - 2)^3 + (x - 2)^4.
    \end{aligned}$$

    Thus,

    $$\boxed{x^4 - 5x^3 + 5x^2 + x + 2 = (x - 2)^4 + 3(x - 2)^3 - (x - 2)^2 - 7(x - 2)}.$$

2. Expand the polynomial $Q(x) = x^5 + 2x^4 - x^2 + x + 1$ in powers of $x + 1$.

    Expanding around $a = -1$:

    $$Q(x) = \sum_{k=0}^{5} \frac{Q^{(k)}(-1)}{k!}(x + 1)^k.$$

The derivatives are:

$$Q(x) = x^5 + 2x^4 - x^2 + x + 1,$$
$$Q'(x) = 5x^4 + 8x^3 - 2x + 1,$$
$$Q''(x) = 20x^3 + 24x^2 - 2,$$
$$Q'''(x) = 60x^2 + 48x,$$
$$Q^{(4)}(x) = 120x + 48,$$
$$Q^{(5)}(x) = 120.$$

Evaluating at $x = -1$:

$$Q(-1) = 0,$$
$$Q'(-1) = 0,$$
$$Q''(-1) = 2,$$
$$Q'''(-1) = 12,$$
$$Q^{(4)}(-1) = -72,$$
$$Q^{(5)}(-1) = 120.$$

Hence,

$$Q(x) = \frac{0}{0!} + \frac{0}{1!}(x+1) + \frac{2}{2!}(x+1)^2 + \frac{12}{3!}(x+1)^3 + \frac{-72}{4!}(x+1)^4 + \frac{120}{5!}(x+1)^5$$
$$= (x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5.$$

Thus,

$$\boxed{x^5 + 2x^4 - x^2 + x + 1 = (x+1)^5 - 3(x+1)^4 + 2(x+1)^3 + (x+1)^2}$$

Exercise 3: Taylor's Formula for $f(x) = \sqrt{x}$ at $a = 1$, $n = 3$

Taylor's formula with Lagrange remainder for a function f that is $n + 1$ times differentiable is:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1},$$

and ξ lies between a and x.

For $f(x) = \sqrt{x} = x^{1/2}$, $a = 1$, and $n = 3$, we compute the derivatives:

$$f(x) = x^{1/2}, \qquad\qquad f(1) = 1,$$

$$f'(x) = \frac{1}{2}x^{-1/2}, \qquad\qquad f'(1) = \frac{1}{2},$$

$$f''(x) = -\frac{1}{4}x^{-3/2}, \qquad\qquad f''(1) = -\frac{1}{4},$$

$$f'''(x) = \frac{3}{8}x^{-5/2}, \qquad\qquad f'''(1) = \frac{3}{8},$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}.$$

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The Taylor polynomial of degree 3 is:

$$P_3(x) = 1 + \frac{1}{2}(x-1) + \frac{-\frac{1}{4}}{2!}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

The remainder term in Lagrange form is:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-1)^4 = \frac{-\frac{15}{16}\xi^{-7/2}}{24}(x-1)^4 = -\frac{5}{128}\frac{(x-1)^4}{\xi^{7/2}},$$

where ξ is a number between 1 and x. Equivalently, we can write $\xi = 1 + \theta(x-1)$ with $0 < \theta < 1$.
Thus, the complete Taylor formula for $\sqrt{x}$ is:

$$\boxed{\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}\frac{(x-1)^4}{\big(1+\theta(x-1)\big)^{7/2}}, \qquad 0 < \theta < 1.}$$

Exercise 4: Maclaurin Formula for $f(x) = \sqrt{1+x}$

(1) Give the Maclaurin formula for $f(x) = \sqrt{1+x}$ when $n = 2$.

The Maclaurin formula (Taylor expansion at $a = 0$) with Lagrange remainder is

$$f(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} x^k + R_2(x),$$

where

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!} x^3,$$

and ξ lies between 0 and x (equivalently $\xi = \theta x$ with $0 < \theta < 1$).

Compute the derivatives:

$$f(x) = (1+x)^{1/2}, \qquad\qquad f(0) = 1,$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \qquad\qquad f'(0) = \frac{1}{2},$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}, \qquad\qquad f''(0) = -\frac{1}{4},$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}.$$

Hence,

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{-\frac{1}{4}}{2!}x^2 + R_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + R_2(x),$$

with

$$R_2(x) = \frac{1}{3!}\frac{3}{8}(1+\xi)^{-5/2}x^3 = \frac{1}{16}(1+\xi)^{-5/2}x^3, \qquad 0 < \xi < x.$$

Writing $\xi = \theta x$ ($0 < \theta < 1$), we obtain the formula

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}(1+\theta x)^{-5/2}x^3, \qquad 0 < \theta < 1.}$$

(2) Give the error of the approximation $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ when $x = 0.2$.

The error is exactly the remainder $R_2(0.2)$. For $x = 0.2$,

$$R_2(0.2) = \frac{1}{16}(1+0.2\theta)^{-5/2}(0.2)^3 = \frac{0.008}{16}(1+0.2\theta)^{-5/2} = 0.0005\,(1+0.2\theta)^{-5/2},$$

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with $0 < \theta < 1$.

The factor $(1 + 0.2\theta)^{-5/2}$ is a decreasing function of θ. Therefore,

$$(1 + 0.2)^{-5/2} < (1 + 0.2\theta)^{-5/2} < 1,$$

i.e.,

$$0.634 \approx (1.2)^{-5/2} < (1 + 0.2\theta)^{-5/2} < 1.$$

Consequently,

$$0.0005 \times 0.634 < R_2(0.2) < 0.0005,$$

or

$$3.17 \times 10^{-4} < R_2(0.2) < 5.00 \times 10^{-4}.$$

Using the exact value $\sqrt{1.2} \approx 1.095445$ and the approximation $1 + 0.1 - 0.005 = 1.095$, the actual error is approximately

$$1.095445 - 1.095 = 0.000445.$$

Thus, for $x = 0.2$, the error of the given approximation is about

$$\boxed{4.45 \times 10^{-4}}.$$

(Equivalently, one can say that the error lies between 3.17×10^{-4} and 5.00×10^{-4}.)

Exercise 5: Taylor Expansion of Sine

(1) Show that

$$\sin x = \sin a + (\cos a)(x - a) - \frac{(\sin a)(x - a)^2}{2!} - \frac{(\cos \xi)(x - a)^3}{3!}, \qquad a < \xi < x.$$

Solution. *The Taylor expansion of a function f about the point a with Lagrange remainder is*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(\xi)}{3!}(x - a)^3,$$

where ξ lies between a and x. For $f(x) = \sin x$ we have

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x.$$

Substituting,

$$\sin x = \sin a + \cos a\,(x - a) + \frac{-\sin a}{2!}(x - a)^2 + \frac{-\cos \xi}{3!}(x - a)^3,$$

which is exactly the required formula.

(2) Use the formula to approximate $\sin 49°$ and estimate the error.

Solution. *Choose $a = 45°$ (a point where sine and cosine are known exactly) and $x = 49°$. Convert to radians:*

$$a = 45° = \frac{\pi}{4}, \qquad x = 49° = \frac{49\pi}{180}, \qquad x - a = 4° = \frac{\pi}{45}.$$

Also,

$$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}.$$

Using the expansion up to the quadratic term (neglecting the remainder), we obtain

$$\sin 49° \approx \sin 45° + \cos 45° \cdot \frac{\pi}{45} - \frac{\sin 45°}{2!}\left(\frac{\pi}{45}\right)^2$$

$$= \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{45} - \frac{1}{2}\left(\frac{\pi}{45}\right)^2\right).$$

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Numerical evaluation:

$$\frac{\pi}{45} \approx 0.06981317, \qquad \left(\frac{\pi}{45}\right)^2 \approx 0.004873, \qquad \frac{1}{2}\left(\frac{\pi}{45}\right)^2 \approx 0.0024365.$$

Hence,

$$1 + \frac{\pi}{45} - \frac{1}{2}\left(\frac{\pi}{45}\right)^2 \approx 1.06737667,$$

and

$$\sin 49^\circ \approx 0.70710678 \times 1.06737667 \approx 0.7547.$$

(The exact value to six decimals is $\sin 49^\circ \approx 0.754710$.*)*

The error of this approximation is given by the Lagrange remainder:

$$R = -\frac{\cos \xi}{6}\left(\frac{\pi}{45}\right)^3, \qquad 45^\circ < \xi < 49^\circ.$$

Since $\cos$ *is positive and decreasing on* $[45^\circ, 49^\circ]$, *the maximum of* $|\cos \xi|$ *occurs at* $\xi = 45^\circ$:

$$|\cos \xi| \leq \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

Therefore,

$$|R| \leq \frac{\sqrt{2}/2}{6}\left(\frac{\pi}{45}\right)^3 = \frac{\sqrt{2}}{12}\left(\frac{\pi}{45}\right)^3.$$

Compute:

$$\left(\frac{\pi}{45}\right)^3 \approx (0.06981317)^3 \approx 0.0003401, \qquad \frac{\sqrt{2}}{12} \approx 0.11785,$$

so

$$|R| \lesssim 0.11785 \times 0.0003401 \approx 4.01 \times 10^{-5} < 5 \times 10^{-5}.$$

Hence the absolute error is smaller than 0.00005.

Exercise 6: Maclaurin Series of $e^x \sin x$

The Maclaurin series expansion of a function $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n.$$

For $f(x) = e^x \sin x$, one can show (by induction) that

$$f^{(n)}(x) = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right).$$

Hence,

$$f^{(n)}(0) = (\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right).$$

Therefore, the Maclaurin series is

$$\boxed{e^x \sin x = \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right)}{n!}\, x^n.}$$

Explicitly computing the first few coefficients:

$$n = 0 : \sin 0 = 0,$$
$$n = 1 : \sqrt{2}\sin\frac{\pi}{4} = 1,$$
$$n = 2 : 2\sin\frac{\pi}{2} = 2,$$
$$n = 3 : 2\sqrt{2}\sin\frac{3\pi}{4} = 2,$$
$$n = 4 : 4\sin\pi = 0,$$
$$n = 5 : 4\sqrt{2}\sin\frac{5\pi}{4} = -4,$$
$$n = 6 : 8\sin\frac{3\pi}{2} = -8,$$
$$n = 7 : 8\sqrt{2}\sin\frac{7\pi}{4} = -8, \text{ etc.}$$

Dividing by the corresponding factorials gives the series

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \frac{1}{630}x^7 + \cdots .$$

(One may also obtain this series by multiplying the Maclaurin series of e^x and $\sin x$.)

Exercise 7: Inequalities via Taylor Expansions

(a) **Maclaurin expansion of order n for e^x**

The Maclaurin expansion of $f(x) = e^x$ is given by:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}, \quad \text{with } 0 < \theta < 1.$$

For $x \geq 0$, the remainder term is non-negative:

$$\frac{x^{n+1}}{(n+1)!}e^{\theta x} \geq 0.$$

Therefore,

$$e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

(b) **Approximation for $n = 2$ at $x = 1$**

For $n = 2$, the expansion is:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{6}e^{\theta x}, \quad \text{with } 0 < \theta < 1.$$

Setting $x = 1$, we obtain:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6}e^\theta = \frac{5}{2} + \frac{1}{6}e^\theta.$$

Since $0 < \theta < 1$, we know that $1 < e^\theta < e$. We apply this to the equation:

$$\frac{5}{2} + \frac{1}{6}(1) < \frac{5}{2} + \frac{1}{6}e^\theta < \frac{5}{2} + \frac{1}{6}e.$$

Substituting e back into the middle term:

$$\frac{5}{2} + \frac{1}{6} < e < \frac{5}{2} + \frac{1}{6}e.$$

From the left side:
$$e > \frac{15}{6} + \frac{1}{6} = \frac{16}{6} = \frac{8}{3}.$$

From the right side:
$$e < \frac{5}{2} + \frac{1}{6}e \implies \frac{5}{6}e < \frac{5}{2} \implies e < 3.$$

Thus, we conclude:
$$\boxed{\frac{8}{3} < e < 3}.$$

(c) **General estimation for** $x = 1$

Using the expansion for $x = 1$ at order n:
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}, \quad \text{with } 0 < \theta < 1.$$

Rearranging for the remainder:
$$e - \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right] = \frac{e^\theta}{(n+1)!}.$$

Using the bounds $1 < e^\theta < e < 3$ established previously:
$$\frac{1}{(n+1)!} < \frac{e^\theta}{(n+1)!} < \frac{3}{(n+1)!}.$$

Therefore:
$$\frac{1}{(n+1)!} < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}.$$

Question 2: Further Maclaurin Applications

1) **Maclaurin formula of order 4 for** $\cos(x)$

$$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} \sin(\theta x), \quad \text{with } 0 < \theta < 1.$$

Consider $x \in [0, \frac{\pi}{2}]$. Then $\theta x \in [0, \frac{\pi}{2}]$, which implies $\sin(\theta x) \geq 0$. Consequently, the remainder term is non-positive:
$$-\frac{x^5}{5!} \sin(\theta x) \leq 0.$$

Thus:
$$\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

2) **Inequalities for** $\ln(x+1)$

We examine the expansions for orders $n = 3$ and $n = 4$ for $x > 0$.

Order $n = 3$:
$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(\theta x + 1)^4}.$$

Since $x > 0$, the remainder term is negative:
$$-\frac{x^4}{4(\theta x + 1)^4} < 0 \implies \ln(x+1) < x - \frac{x^2}{2} + \frac{x^3}{3}.$$

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Order $n = 4$:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5(\theta x + 1)^5}.$$

Since $x > 0$, the remainder term is positive:

$$\frac{x^5}{5(\theta x + 1)^5} > 0 \implies \ln(x+1) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

Combining these results, for $x > 0$:

$$\boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} < \ln(x+1) < x - \frac{x^2}{2} + \frac{x^3}{3}.}$$

Exercise 8: Special Relativity Kinetic Energy

1. **Classical Limit:**

   The relativistic kinetic energy is given by:

   $$K = m_0 c^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right].$$

   Using the binomial or Taylor expansion $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \mathcal{O}(x^2)$ with $x = v^2/c^2$:

   $$K = m_0 c^2 \left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} \right) - 1 \right] = m_0 c^2 \left(\frac{v^2}{2c^2} \right).$$

   Simplifying yields the classical Newtonian kinetic energy:

   $$\boxed{K \approx \frac{1}{2} m_0 v^2.}$$

2. **Error Estimation (Taylor's Formula):**

   Let $f(x) = (1-x)^{-1/2}$ where $x = v^2/c^2$. The Taylor expansion of order 1 with the Lagrange remainder is:

   $$f(x) = 1 + \frac{1}{2}x + \frac{f''(\xi)}{2!}x^2, \quad \text{where } 0 < \xi < x.$$

   Calculating the second derivative, $f''(x) = \frac{3}{4}(1-x)^{-5/2}$. The difference between the relativistic and classical energy (ΔK) corresponds to the remainder term:

   $$\Delta K = m_0 c^2 \left(\frac{f''(\xi)}{2!}x^2 \right) = m_0 c^2 \left(\frac{3}{8}(1-\xi)^{-5/2} \left(\frac{v^2}{c^2} \right)^2 \right).$$

   $$\Delta K = \frac{3}{8} m_0 \frac{v^4}{c^2}(1-\xi)^{-5/2}.$$

   Numerical Evaluation: Given $|v| \leq 100$ m/s and $c \approx 3 \times 10^8$ m/s, we have $x = (v/c)^2 \approx 1.1 \times 10^{-13}$. Since $0 < \xi < x \ll 1$, we approximate $(1-\xi)^{-5/2} \approx 1$.

   $$\Delta K \approx \frac{3}{8} m_0 \frac{(100)^4}{(3 \times 10^8)^2} = \frac{3}{8} m_0 \frac{10^8}{9 \times 10^{16}}.$$

   $$\Delta K \approx 4.17 \times 10^{-10} \, m_0 \text{ J.}$$

   This discrepancy is negligible for macroscopic objects at non-relativistic speeds.