Chapter 4: Series expansion and Taylor series

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Plan

- References
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- Mean Value Theorems
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References

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References

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1. Comparison relations

Introduction

In mathematical analysis, we frequently encounter functions that approach zero as a variable tends to a specific value. Understanding their relative rates of decay is essential for studying asymptotic behavior. Landau notations (e.g., Big-O and little-o) provide a formal framework to describe these relationships, closely tied to concepts in infinitesimal analysis such as orders and equivalence. Consider multiple infinitesimal quantities $\alpha, \beta, \gamma, \ldots$ which depend on the same variable x and tend towards zero as x approaches a certain limit, either α or infinity. The manner in which these variables approach zero will be examined when we analyze their ratios. We will adopt the following definitions.

Definitions

Definition 1:

- If the ratio $\frac{\beta}{\alpha}$ has a finite non-zero limit, denoted by A, meaning $\lim_{x\to 0} \frac{\beta}{\alpha} = A \neq 0$, and consequently $\lim_{x\to 0} \frac{\alpha}{\beta} = \frac{1}{A} \neq 0$, then the infinitesimals β and α are known as **infinitesimals of the same order**.
- **Connection to Big-O (Landau notation)**
  This implies $\beta = O(\alpha)$ and $\alpha = O(\beta)$, meaning neither function dominates the other asymptotically.

Example 1:

- Let us take $\alpha = x$ and $\beta = \sin(2x)$, then we have

$$\lim_{x \to 0} \frac{\beta}{\alpha} = \lim_{x \to 0} \frac{\sin 2x}{x} = 2.$$

- So α and β are of the same order where $x \to 0$.
- And $\sin(2x) = O(x)$ and $x = O(\sin(2x))$.

Example 2:

- As x approaches zero, the quantities x, $\sin(3x)$, $\tan(2x)$, and $7\ln(1+x)$ exhibit similar behavior, indicating they are of the same order.

Definition 2:

- When the ratio of two infinitesimals α and β approaches zero; which means $\lim_{x\to 0} \frac{\beta}{\alpha} = 0$ (or $\lim_{x\to 0} \frac{\alpha}{\beta} = \infty$), the infinitesimal β is considered of higher order than the infinitesimal α, while α is of lower order than β.
- **Connection to Little-o (Landau notation)**
  This corresponds to $\beta = o(\alpha)$, indicating β approaches zero faster than α.

Example 3:

- Given $\alpha = x$ and $\beta = x^n$, where $n > 1$ and $x \to 0$, β is of a higher order compared to α since x^n approaches zero faster than x; so that we have $\lim_{x\to 0} \frac{\beta}{\alpha} = \lim_{x\to 0} \frac{x^n}{x} = 0$. We can also say α is of a lower order compared to β.
- Thus $x^n = o(x)$.

Definition 3:

- An infinitesimal β is considered of the k th order relative to another infinitesimal α if the infinitesimals $\frac{\beta}{\alpha^k}$ and β are of the same order.

Example 4:

- With $\alpha = x$ and $\beta = x^4$, β is a forth-order infinitesimal relative to α; since $\frac{x^4}{x^4}$ approaches zero faster than x, that is, $\lim_{x \to 0} \frac{\beta}{\alpha^4} = \lim_{x \to 0} \frac{x^4}{x^4} = 1$.

Definition 4:

- Let α and β be two infinitesimals. We say that α is equivalent to β as x tends to x_0, denoted by $\alpha \sim \beta$, if $\lim_{x \to x_0} \frac{\beta}{\alpha} = 1$.
- **Representation Using Little-o**
  This is expressed as $\beta = \alpha + o(\alpha)$, where the difference is of higher order than α.

Definition 4: Equivalence
$\alpha \sim \beta$ if $\lim_{x\to 0} \frac{\beta}{\alpha} = 1$.

Representation Using Little-o
This is expressed as $\beta = \alpha + o(\alpha)$, where the difference is of higher order than α.

Example 5:

- For $\alpha = x$ and $\beta = \sin x$,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

  so $\sin x = x + o(x)$.

- As $x \to 0$ we have $\tan x \sim x$, $e^x - 1 \sim x$, $\ln(x+1) \sim x$, $1 - \cos x \sim \frac{x^2}{2}$.

Theorems

Theorem 1:

- When comparing two infinitesimals α and β, if they are equivalent, their difference $\alpha - \beta$ is an infinitesimal with a higher order than both of them.

Proof of Theorem 1

Proof:

- By evaluating the limit

$$\lim_{x\to x_0}\frac{\alpha-\beta}{\alpha}=\lim_{x\to x_0}\left(1-\frac{\beta}{\alpha}\right)=1-1=0.$$

Theorems

Theorem 2:

- If the difference between two infinitesimals $\alpha - \beta$ is an infinitesimal of higher order than both α and β are equivalent infinitesimals.

Proof of Theorem 2 (Continued)

Proof:

- If $\lim_{x\to x_0} \frac{\alpha-\beta}{\alpha} = 0 \Leftrightarrow 1 - \lim_{x\to x_0} \frac{\beta}{\alpha} = 0$ so $1 = \lim_{x\to x_0} \frac{\beta}{\alpha}$. Thus we have $\alpha \sim \beta$.

Example 7:

- Consider the infinitesimals $\alpha = x$ and $\beta = x + x^2$ where x approaches zero. Then we have

$$\lim_{x\to 0} \frac{\beta - \alpha}{\alpha} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

and

$$\lim_{x\to 0} \frac{\alpha - \beta}{\beta} = \lim_{x\to 0} \frac{-x^2}{x + x^2} = \lim_{x\to 0} \frac{-x}{1 + x} = 0$$

Note

Note:

- If the ratio of two infinitesimals has no limit and does not approach infinity, then they are not comparable in the given context.

Theorems Connecting Equivalence and Little-o

Theorem 1
If $\alpha \sim \beta$, then $\alpha - \beta = o(\alpha)$.

Theorem 2
If $\alpha - \beta = o(\alpha)$, then $\alpha \sim \beta$.

Example 8
For $\alpha = x$ and $\beta = x + x^2$,

$$\lim_{x \to 0} \frac{x + x^2 - x}{x} = \lim_{x \to 0} \frac{x^2}{x} = 0,$$

since $x^2 = o(x)$, thus $x \sim x + x^2$.

Equivalent of some known functions in vicinity of zero:

Table 1:

Function	Equivalent in vicinity of zero
$\sin(x)$	x
$1-\cos(x)$	$\frac{x^2}{2}$
$\tan(x)$	x
$\ln(1+x)$	x
$(1+x)^a$	$1+ax$
$(1-x)^a$	$1-ax$

Table of Common Equivalents in Landau Notation

Function	**Landau Form as** $x \to 0$
$\sin(x)$	$x + o(x)$
$1 - \cos(x)$	$\frac{x^2}{2} + o(x^2)$
$\ln(1 + x)$	$x + o(x)$
$\tan(x)$	$x + o(x)$

Table: Common functions and their Landau equivalents near 0.

2. Mean Value Theorems

Mean Value Theorems

The Mean Value Theorems describe fundamental properties of differentiable functions.

Rolle's Theorem

Statement: If a function $f(x)$ satisfies the following conditions:

- It is continuous on the closed interval $[a, b]$.
- It is differentiable on the open interval (a, b).
- It has equal values at the endpoints: $f(a) = f(b)$.

Then, there exists a point $\xi \in (a, b)$ such that:

$$f'(\xi) = 0. \tag{1}$$

This means that the function has at least one stationary point where the tangent is horizontal.

Mean Value Theorem (MVT)

Statement: If a function $f(x)$ is:

- Continuous on $[a, b]$,
- Differentiable on (a, b),

then there exists at least one point $\xi \in (a, b)$ where:

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \quad (2)$$

This means that at some point inside the interval, the function's instantaneous rate of change equals its average rate of change.

Alternative Forms of MVT

The theorem can also be written in different forms:

- If x and x_0 are in (a, b), then:

$$f(x) = f(x_0) + f'(\xi)(x - x_0), \quad \xi \text{ between } x_0 \text{ and } x. \tag{3}$$

- If we write $b = a + h$, then:

$$\xi = a + \theta h, \quad 0 < \theta < 1. \tag{4}$$

This theorem is also called the **law of the mean**.

Cauchy's Mean Value Theorem (Generalized MVT)

Statement: If two functions $f(x)$ and $g(x)$ satisfy:

- They are continuous on $[a, b]$.
- They are differentiable on (a, b).
- Their values at the endpoints are not equal: $g(a) \neq g(b)$.

Then, there exists some point $\xi \in (a, b)$ such that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{5}$$

If we take $g(x) = x$, we recover the standard Mean Value Theorem.

3. Taylor's Formula (or Taylor's theorem of the mean.)

Taylor's Formula (or Taylor's theorem of the mean.)

Expressing functions as infinite polynomials, known as power series, is a valuable technique. Polynomial functions simplify analysis as they only entail basic arithmetic operations. If a complex function can be represented as an infinite polynomial, we can utilize this representation for differentiation, integration, and approximation. Thus, the crucial inquiry arises: under what conditions can a function be represented by a power series? To delve into this question, let's revisit the concept of geometric series.

Illustrative example:

- Let us consider the following geometric series:

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

- We recall that

$$a + ar + ar^2 + ar^3 + \cdots$$

  is convergent if and only if $|r| < 1$. In our case, we have

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- So in this example, we have written the function $f(x) = \frac{1}{1-x}$ as infinite polynomial

Illustrative Example Figure

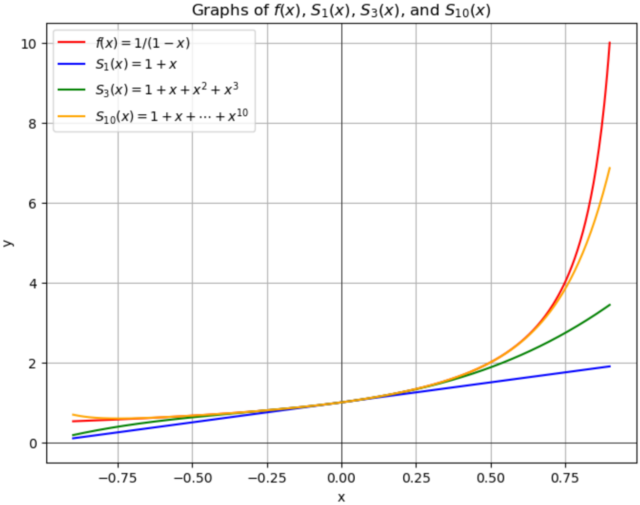


Figure: The above Fig represents the graphs of $f(x) = \frac{1}{1-x}$, $S_1(x) = 1 + x$, $S_3(x) = 1 + x + x^2 + x^3$, and $S_{10}(x) = 1 + x + \cdots + x^{10}$.

2. An overview on Taylor's formula:

- Let's suppose that the function $y = f(x)$ possesses derivatives up to the $(n+1)$th order within an interval containing $x = a$. We aim to find a polynomial $y = P_n(x)$ of degree not exceeding n, such that at $x = a$, its value matches $f(a)$ and its derivatives up to the nth order match those of $f(x)$ at that point: $P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \ldots, P_n^{(n)}(a) = f^{(n)}(a)$. It is reasonable to assume that such a polynomial approximates $f(x)$ closely. We seek this polynomial in the form of a polynomial in terms of $(x - a)$ with coefficients to be determined. We determine the coefficients $C_1, C_2, \ldots, C_n$ to satisfy the given conditions.

Let's start by finding the derivatives of the polynomial $P_n(x)$ with respect to x:

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots + nC_n(x-a)^{n-1}$$
$$P_n''(x) = 2C_2 + 3\cdot 2C_3(x-a) + \cdots + n(n-1)C_n(x-a)^{n-2}$$

Deriving Coefficients

By substituting the value of a for x and replacing $P_n'(a)$ with $f'(a)$, etc., we derive:

$$f(a) = C_0$$
$$f'(a) = C_1$$

Deriving Coefficients (Continued)

$$f''(a) = 2 \cdot 1 \cdot C_2$$
$$f'''(a) = 3 \cdot 2 \cdot 1 \cdot C_3$$
$$\cdots$$
$$f^{(n)}(a) = n(n-1)(n-2) \cdots 2 \cdot 1 \cdot C_n$$

From which we obtain:

$$C_0 = f(a)$$
$$C_1 = f'(a)$$
$$C_2 = \frac{f''(a)}{2!}$$
$$\cdots$$
$$C_n = \frac{f^{(n)}(a)}{n!}$$

Taylor Polynomial $P_n(x)$

Substituting the found values of $C_1, C_2, \ldots, C_n$ into the polynomial $P_n(x)$, we obtain:

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

Remainder Term $R_n(x)$

Let $R_n(x)$ denote the difference between the function $f(x)$ and the constructed polynomial $P_n(x)$:

$$R_n(x) = f(x) - P_n(x)$$

Therefore:

$$f(x) = P_n(x) + R_n(x)$$

Expanding this expression, we get:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \tag{6}$$

Remainder and Taylor's Theorem

- The term $R_n(x)$ is called the remainder. For those values of x where the remainder $R_n(x)$ is small, the polynomial $P_n(x)$ yields an approximate representation of the function $f(x)$. Thus, we can replace the function $y = f(x)$ by the polynomial $y = P_n(x)$ to an appropriate degree of accuracy equal to the value of the remainder $R_n(x)$.
- The Remainder can be written, for example, in the following forms:

Lagrange's and Cauchy's Form of Remainder

Lagrange's form:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \tag{7}$$

Cauchy's form:

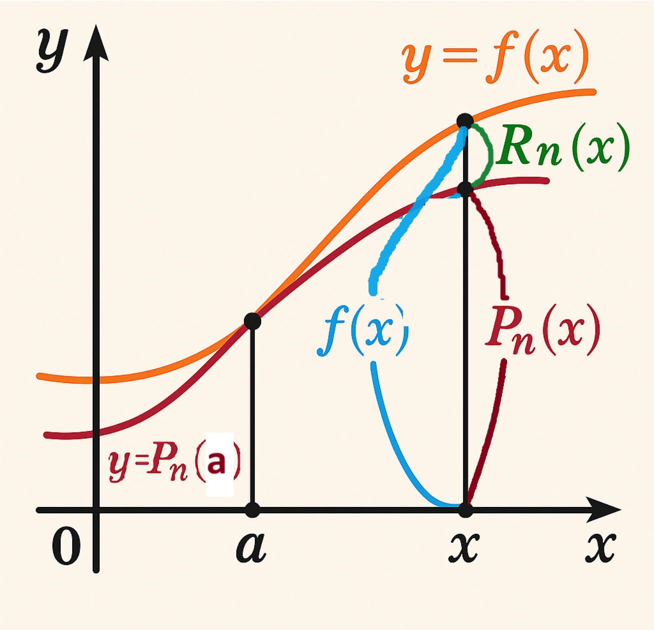
$$R_n = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a) \tag{8}$$

where $a < \xi < x$.

Understanding the Remainder $R_n(x)$

- $R_n(x)$ is called the remainder. For all values of x for which the remainder is small, the polynomial $P_n(x)$ gives a reasonably good approximation of the function $f(x)$.
- Thus, formula (6) allows us to replace the function $y = f(x)$ by the polynomial $y = P_n(x)$ with a degree of precision equal to the remainder $R_n(x)$.
- The problem that now arises is to evaluate the remainder $R_n(x)$ for different values of x.

Figure



Expressing the Remainder

Let's write the remainder in the form:

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} Q(x), \tag{9}$$

where $Q(x)$ is a function to be determined.

Let's put formula (6) in the form:

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}Q(x). \qquad (10)$$

Introducing Auxiliary Function $F(t)$

For x and a fixed, the function $Q(x)$ has a well-determined value; let's denote it by Q.
Consider then an auxiliary function of t (where t is between a and x):

$$F(t) = f(x) - f(t) - \frac{x-t}{1!}f'(t) - \frac{(x-t)^2}{2!}f''(t) - \cdots - \frac{(x-t)^n}{n!}f^{(n)}(t) - \frac{(x-t)^{n+1}}{(n+1)!}Q,$$

where Q is defined by the relation (10); we assume that a and x are well-determined numbers.

Calculating the Derivative $F'(t)$

Let's calculate the derivative $F'(t)$:

$$
\begin{aligned}
F'(t) = & -f'(t) + f'(t) - \frac{x-t}{1!} f''(t) + \frac{2(x-t)}{2!} f''(t) - \cdots \\
& - \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) + \frac{n(x-t)^{n-1}}{n!} f^{(n)}(t) \\
& - \frac{(x-t)^{n}}{n!} f^{(n+1)}(t) + \frac{(n+1)(x-t)^{n}}{(n+1)!} Q
\end{aligned}
$$

Simplified Derivative $F'(t)$

or after simplification:

$$F'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t) + \frac{(x-t)^n}{n!} Q \qquad (11)$$

Existence of $F'(t)$ and Conditions for Rolle's Theorem

- Thus, the derivative of the function $F(t)$ exists for all points t in the vicinity of the point of abscissa a ($a \leq t \leq x$ for $a < x$ and $a \geq t \geq x$ for $a > x$).
- Note also that [by virtue of formula (10)]

$$F(x) = 0, \quad F(a) = 0.$$

- Therefore, the conditions for the validity of Rolle's theorem are met for the function $F(t)$, and consequently, there exists a value $t = \xi$, comprised between a and x, for which $F'(\xi) = 0$.

Applying Rolle's Theorem and Solving for Q

Using relation (11) with $F'(\xi) = 0$:

$$-\frac{(x-\xi)^n}{n!} f^{(n+1)}(\xi) + \frac{(x-\xi)^n}{n!} Q = 0,$$

from which we deduce:

$$Q = f^{(n+1)}(\xi).$$

Lagrange's Form of the Remainder

By substituting this expression for Q in formula (9) we have:

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

This is Lagrange's formula for the remainder.

Expressing ξ and Final Lagrange Remainder Formula

- Since ξ is comprised between x and a, we can put it in the form:

$$\xi = a + \theta(x - a),$$

where θ is a number comprised between 0 and 1, that is to say $0 < \theta < 1$; the formula which gives the remainder becomes:

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}[a + \theta(x - a)].$$

MacLaurin's Formula

When $a = 0$, the Taylor expansion given is referred to as *MacLaurin's formula*, with the Lagrange form of the remainder. In this specific case, the expansion can be written as:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},$$

where ξ is a point in the interval $[0, x]$.

Taylor–Young Formula

We relax the assumptions of Taylor-Lagrange formula and suppose, in the following theorem, that only $f^{(n)}(a)$ exists. Nevertheless, note that the existence of $f^{(n)}(a)$ still requires f to be defined on an interval centered at a, that f admits derivatives up to order $(n-1)$ on that interval, and that $f^{(n-1)}$ is slightly more than just a simple function. Consequently, the resulting expansion is somewhat "weaker" in terms of polynomial approximation. In fact, the remainder $R_n(x)$ is less precise than Lagrange's form or Cauchy's form. All that can be stated is:

$$R_n(x) = (x-a)^n\,\varepsilon(x),$$

where

$$\lim_{x \to a} \varepsilon(x) = 0.$$

That is equivalent to:

$$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0.$$

Taylor–Young Formula

- When we want to approximate a function $f(x)$ near a point a, we can use a polynomial that is built from the function's values and its derivatives at a. This is similar to the well-known Taylor series. However, in the Taylor–Young formula, we relax some of the conditions that are usually needed.
- **Relaxing the Conditions:** In the usual Taylor series, we require that the function has derivatives of all orders (or at least up to a high order) in a neighborhood around a. In the Taylor–Young version, we assume only that the nth derivative $f^{(n)}(a)$ exists. Even though we assume the existence of the nth derivative, the function must still be defined in an interval around a and have its lower-order derivatives up to $n-1$.

- **What Does This Mean for the Approximation?**
  Because we are making fewer assumptions (we don't assume as much smoothness), the polynomial approximation we get is "weaker" or less precise than the full Taylor series approximation. In other words, we can still write the function as a polynomial of degree n plus an error term (called the remainder), but we have less information about the exact size of that error.
- The formula tells us that the remainder $R_n(x)$ can be written as:

$$R_n(x) = (x-a)^n\,\varepsilon(x),$$

where $\varepsilon(x)$ is some function that satisfies:

$$\lim_{x\to a} \varepsilon(x) = 0.$$

This means that as x gets closer and closer to a, the function $\varepsilon(x)$ becomes very small.

- The fact that
$$\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0$$
tells us that the error $R_n(x)$ is of a smaller order than $(x-a)^n$. In simpler terms, when x is very near a, the error term shrinks much faster than the main polynomial term $(x-a)^n$. This assures us that the polynomial is a good approximation of $f(x)$ near a.

- **Why Is This Useful?**
  Even when we don't have all the derivatives needed for a full Taylor series, the Taylor–Young formula still allows us to approximate the function near a. It provides a way to understand how the function behaves close to that point, with the knowledge that the error in the approximation becomes negligible as we approach a.

Theorem
Let $f : I_n =]a-\eta, a+\eta[\to \mathbb{R}$. Suppose that $f^{(n)}(a)$ exists. Then, for all $x \in I_n$, we have:

$$f(x) \;=\; f(a) + f'(a)\,(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n + (x-a)^n\,\varepsilon(x), \tag{12}$$

where

$$\lim_{x\to a} \varepsilon(x) \;=\; 0.$$

Formula (12) is called the **Taylor–Young Formula**.
Remark: If $a = 0$, then formula (12) is called the **Maclaurin–Young Formula**, and it takes the following form:

$$f(x) \;=\; f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(n)}(0)}{n!}\,x^n + x^n\,\varepsilon(x),$$

with

$$\lim_{x\to 0} \varepsilon(x) \;=\; 0.$$

Rule 2:

- If f and g are two functions defined in the neighborhood of $x = 0$ (except possibly at $x = 0$) and admit series expansions to the order n in the neighborhood of 0.
- Let A and B be their regular parts respectively.

a) Sum:

- $(f+g)(x) = A(x) + B(x) + x^n \varepsilon_3(x)$

Example 1: Addition of Series

Let $f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

and $g(x) = \sin x = x - \frac{x^3}{6} + o(x^3)$.

Their sum $f(x) + g(x)$ to order 3:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \left(x - \frac{x^3}{6}\right) + o(x^3) = 1 + 2x + \frac{x^2}{2} + o(x^3)$$

Example 2: Subtraction of Series

Using $f(x) = e^x$ and $g(x) = \sin x$ as above:

$$f(x) - g(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(x - \frac{x^3}{6}\right) + o(x^3)$$
$$= 1 + \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

b) Product:

- $(f \cdot g)(x) = C(x) + x^n \varepsilon_4(x)$
- To obtain the regular part C of the product $f \cdot g$, we must keep in the product $A \cdot B$ only the terms whose degree is less than or equal to n.

Example 3: Multiplication of Series

Let $f(x) = e^x \approx 1 + x + \frac{x^2}{2}$ (to x^2)
and $g(x) = \cos x \approx 1 - \frac{x^2}{2}$ (to x^2).
Their product $f(x)g(x)$ to x^2:

$$\left(1 + x + \frac{x^2}{2}\right)\left(1 - \frac{x^2}{2}\right) + o(x^2) = 1 + x + \frac{x^2}{2} - \frac{x^2}{2} + o(x^2) = 1 + x + o(x^2)$$

c) Quotient:

- We show that $(\frac{f}{g})(x) = Q(x) + x^n \varepsilon_5(x)$ where Q is the regular part of the quotient $\frac{A}{B}$ according to the ascending powers of x.

Example 4: Division of Series

Let $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ (to x^3)
and $g(x) = 1 + x$.
The quotient $\frac{f(x)}{g(x)}$ to x^3:

$$\frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6}}{1 + x} = 1 + \frac{x^2}{2} - \frac{x^3}{3} + o(x^3) \quad \text{(via polynomial division)}$$

Limit of Errors

With:

$$\lim_{x \to 0} \varepsilon_3(x) = \lim_{x \to 0} \varepsilon_4(x) = \lim_{x \to 0} \varepsilon_5(x) = 0$$

Rule 3:

- If $f(x) = A(x) + o(x^n)$ and $g(x) = B(x) + o(x^n)$ with $B(0) = 0$

Then:

- $(f \circ g)(x) = (A \circ B)(x) + o(x^n)$
- Where we must keep in $(A \circ B)(x)$ only the terms whose degree is less than or equal to n.

Another expression:

- If:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + x^n \varepsilon_1(x)$$
$$g(x) = b_1 x + b_2 x^2 + \cdots + b_n x^n + x^n \varepsilon_2(x)$$

- Then:

$$\begin{aligned}(f \circ g)(x) &= a_0 + a_1(g(x)) + a_2(g(x))^2 + \cdots \\ &\quad + a_n(g(x))^n + (g(x))^n \varepsilon_1(g(x)) \\ &= a_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + x^n \varepsilon(x)\end{aligned}$$

Example 5: Composition of Series

Let $f(u) = \frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$

and $g(x) = x + x^2$.

The composition $f(g(x))$ to x^3:

$$\frac{1}{1-(x+x^2)} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots$$

Truncating at x^3:

$$1 + x + x^2 + x^2 + 2x^3 + x^3 = 1 + x + 2x^2 + 3x^3 + o(x^3)$$

4. Applications of Taylor's Formula

Approximating Functions with Polynomials

A function $f(x)$ can be represented by its Taylor series expansion at a point a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{13}$$

In practice, we use a finite number of terms to approximate $f(x)$, forming the Taylor polynomial of degree n:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{14}$$

First-Degree Taylor Polynomial

The first-degree Taylor polynomial is given by:

$$T_1(x) = f(a) + f'(a)(x - a). \tag{15}$$

- This represents the **tangent line** to $f(x)$ at $x = a$.
- It is the simplest linear approximation of $f(x)$.
- Near $x = a$, $T_1(x)$ provides a good estimate of $f(x)$.

Example: Approximating e^x

For $f(x) = e^x$ centered at $x = 0$, the Taylor series expansion is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{16}$$

The first-degree Taylor polynomial:

$$T_1(x) = 1 + x. \tag{17}$$

The second-degree Taylor polynomial:

$$T_2(x) = 1 + x + \frac{x^2}{2}. \tag{18}$$

Higher-Degree Approximations

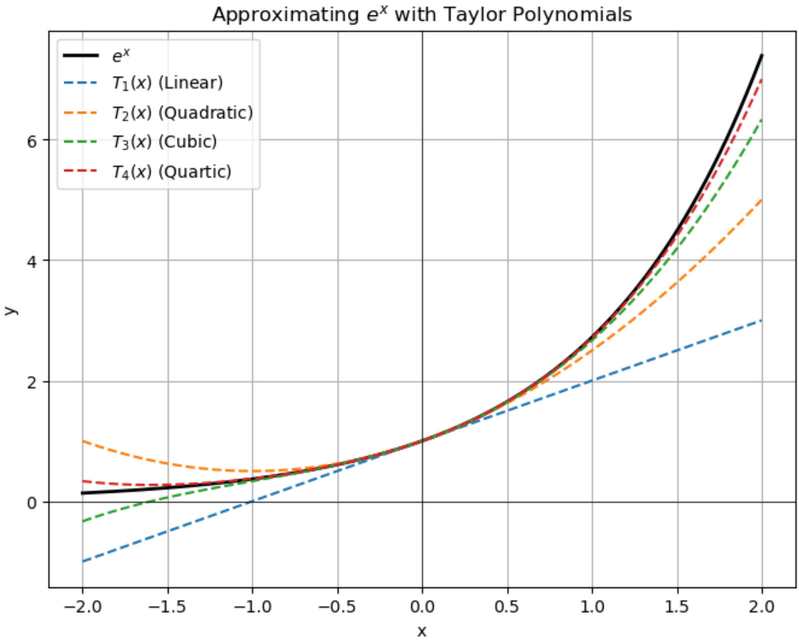
- The third-degree polynomial:

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

- The fourth-degree polynomial:

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

- As n increases, $T_n(x)$ gets closer to e^x.


Approximating e^x with Taylor Polynomials
e^x
$T_1(x)$ (Linear)
$T_2(x)$ (Quadratic)
$T_3(x)$ (Cubic)
$T_4(x)$ (Quartic)
y
x
0
2
4
6
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Finding Limits Using Taylor Series Steps

Taylor series expansions are a powerful tool for calculating limits, especially for indeterminate forms like $\frac{0}{0}$. By expanding functions around a point (often $x = 0$), we can simplify complex expressions and reveal their asymptotic behavior, even with intricate denominators.

Key Takeaways

- Expand all functions to sufficient order, matching the highest power in the denominator or numerator.
- Simplify by canceling or combining terms, especially with complex denominators.
- Retain leading terms to determine the limit's behavior.
- Evaluate the ratio of coefficients or check for divergence.

Taylor series excel at resolving indeterminate forms ($\frac{0}{0}$) and handling intricate function combinations.

Example 1: Basic Trigonometric Limit

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

Solution Steps:

1. Expand $\sin x$ using its Maclaurin series:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

2. Subtract x in the numerator:

$$\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

3. Divide by x^3:

$$\frac{-\frac{x^3}{6} + \frac{x^5}{120} - \cdots}{x^3} = -\frac{1}{6} + \frac{x^2}{120} - \cdots$$

4. Take the limit as $x \to 0$: $-\frac{1}{6}$

Example 2: Exponential and Polynomial

Limit to Evaluate:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Solution Steps:

1. Expand e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

2. Subtract 1 and x:

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

3. Divide by x^2:

$$\frac{\frac{x^2}{2} + \frac{x^3}{6} + \cdots}{x^2} = \frac{1}{2} + \frac{x}{6} + \cdots$$

4. Take the limit as $x \to 0$:

$$\frac{1}{2}$$

Example 3: Composite Functions

Limit to Evaluate:

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

Solution Steps:

1. Expand $\tan x$ and $\sin x$:

$$\tan x = x + \frac{x^3}{3} + \cdots, \quad \sin x = x - \frac{x^3}{6} + \cdots$$

2. Subtract $\sin x$ from $\tan x$:

$$\tan x - \sin x = \left(x + \frac{x^3}{3}\right) - \left(x - \frac{x^3}{6}\right) = \frac{x^3}{2} + \cdots$$

3. Divide by x^3:

$$\frac{\frac{x^3}{2} + \cdots}{x^3} = \frac{1}{2} + \cdots$$

4. Take the limit as $x \to 0$:

$$\frac{1}{2}$$

Limit to Evaluate:

$$\lim_{x \to 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3}$$

Solution Steps:

1. Expand $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

2. Compute the numerator:

$$\ln(1+x) - x + \frac{x^2}{2} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - x + \frac{x^2}{2} = \frac{x^3}{3} - \cdots$$

3. Divide by x^3:

$$\frac{\frac{x^3}{3} - \cdots}{x^3} = \frac{1}{3} - \cdots$$

4. Take the limit as $x \to 0$:

$$\frac{1}{3}$$

Example 5: Higher-Order Terms

Limit to Evaluate:

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

Solution Steps:

1. Expand $\cos x$:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

2. Compute the numerator:

$$\cos x - 1 + \frac{x^2}{2} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - 1 + \frac{x^2}{2} = \frac{x^4}{24} - \cdots$$

3. Divide by x^4:

$$\frac{\frac{x^4}{24} - \cdots}{x^4} = \frac{1}{24} - \cdots$$

4. Take the limit as $x \to 0$:

$$\frac{1}{24}$$

Example 6: Exponential and Logarithmic Denominator

Limit to Evaluate:

$$\lim_{x\to 0} \frac{\sin x - x}{e^x + \ln(1-x) - 1}$$

Solution Steps:

1. Expand the numerator:

$$\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

2. Expand the denominator:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots, \quad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$e^x + \ln(1-x) - 1 = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right) - 1 = -\frac{x^3}{6} - \cdots$$

3. Simplify:

$$\frac{-\frac{x^3}{6} + \cdots}{-\frac{x^3}{6} - \cdots} = 1 + \cdots$$

4. Take the limit as $x \to 0$:

$$1$$

Example 7: Trigonometric-Exponential Denominator

Limit to Evaluate:

$$\lim_{x \to 0} \frac{\tan x - \sin x}{(e^x - 1)(\cos x - 1)}$$

Solution Steps:

1. Expand the numerator:

$$\tan x - \sin x = \frac{x^3}{2} + \cdots$$

2. Expand the denominator:

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots, \quad \cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

$$(e^x - 1)(\cos x - 1) = \left(x + \frac{x^2}{2}\right)\left(-\frac{x^2}{2}\right) + \cdots = -\frac{x^3}{2} - \cdots$$

3. Simplify:

$$\frac{\frac{x^3}{2} + \cdots}{-\frac{x^3}{2} - \cdots} = -1 + \cdots$$

4. Take the limit as $x \to 0$:

$$-1$$

Example 8: Logarithmic-Exponential Combination

Limit to Evaluate:

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{\ln(1+x) - x + \frac{x^2}{2}}$$

Solution Steps:

1. Expand the numerator:

$$e^x - 1 - x - \frac{x^2}{2} = \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

2. Expand the denominator:

$$\ln(1+x) - x + \frac{x^2}{2} = \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

3. Simplify:

$$\frac{\frac{x^3}{6} + \cdots}{\frac{x^3}{3} - \cdots} = \frac{1}{2} + \cdots$$

4. Take the limit as $x \to 0$:

$$\frac{1}{2}$$

Example 9: Nested Functions in the Denominator

Limit to Evaluate:

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{\sin(x^2) - x^2}$$

Solution Steps:

1. Expand the numerator:

$$\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{24} - \cdots$$

2. Expand the denominator:

$$\sin(x^2) = x^2 - \frac{x^6}{6} + \cdots, \quad \sin(x^2) - x^2 = -\frac{x^6}{6} + \cdots$$

3. Simplify:

$$\frac{\frac{x^4}{24} - \cdots}{-\frac{x^6}{6} + \cdots} = \frac{1}{4x^2} + \cdots$$

4. Take the limit as $x \to 0$:

$$\infty \quad \text{(limit diverges)}$$

Example 10: Hyperbolic and Polynomial Denominator

Limit to Evaluate:

$$\lim_{x \to 0} \frac{\sinh x - x}{\cosh x - 1 - \frac{x^2}{2}}$$

Solution Steps:

1. Expand the numerator:

$$\sinh x = x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots, \quad \sinh x - x = \frac{x^3}{6} + \cdots$$

2. Expand the denominator:

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots, \quad \cosh x - 1 - \frac{x^2}{2} = \frac{x^4}{24} + \cdots$$

3. Simplify:

$$\frac{\frac{x^3}{6} + \cdots}{\frac{x^4}{24} + \cdots} = \frac{4}{x} + \cdots$$

4. Take the limit as $x \to 0$:

$$\infty \quad \text{(limit diverges)}$$

Application of Taylor's Formula to Local Optima

Review of Second-Derivative Test

- Let f be twice differentiable at x_0.
- **Sufficient condition**:
  - If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a **local maximum**.
  - If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a **local minimum**.
- **Limitation**: If $f''(x_0) = 0$, the test is inconclusive.

Key Idea

When $f''(x_0) = 0$, analyze higher-order derivatives using Taylor's expansion.

Let n be the smallest integer such that:

$$f^{(n)}(x_0) \neq 0.$$

- If n is **even**: x_0 is a local optimum.
  - Maximum if $f^{(n)}(x_0) < 0$.
  - Minimum if $f^{(n)}(x_0) > 0$.
- If n is **odd**: No local extremum (saddle point).

Theorem (Formal Statement)

Let $f : I_\eta = (x_0 - \eta, x_0 + \eta) \to \mathbb{R}$ satisfy:

1. f is n-times differentiable in I_η.
2. $f^{(n)}$ is continuous in I_η.
3. All lower-order derivatives vanish:

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0.$$

4. $f^{(n)}(x_0) \neq 0$.

Then:

- x_0 is a local optimum **if and only if** n is even.

Proof of Theorem

Taylor's Expansion at $x_0 + h$:

$$f(x_0 + h) = f(x_0) + \frac{f^{(n)}(x_0 + \theta h)}{n!} h^n \quad (0 < \theta < 1).$$

- Subtract $f(x_0)$:

$$f(x_0 + h) - f(x_0) = \frac{f^{(n)}(x_0 + \theta h)}{n!} h^n.$$

- For small h, $f^{(n)}(x_0 + \theta h) \approx f^{(n)}(x_0)$ (by continuity).

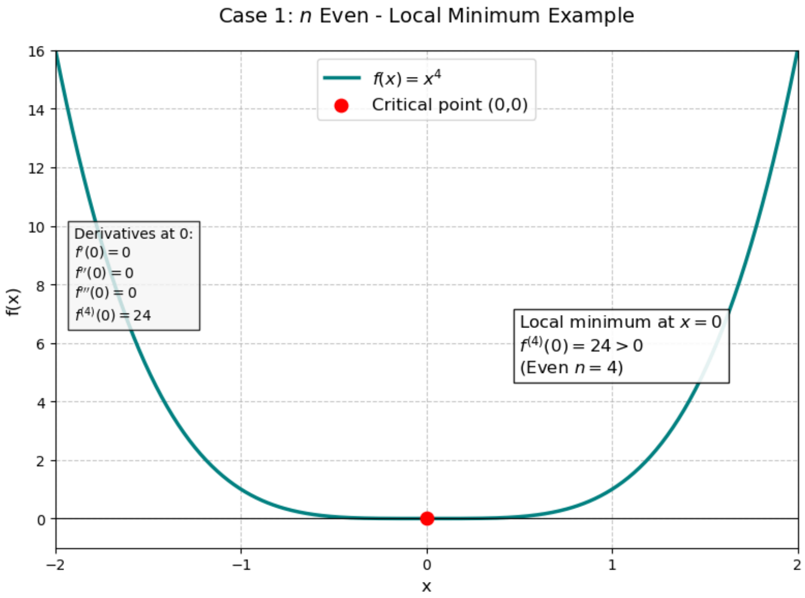
Case 1: n Even

- If n is even: $h^n > 0$ for all $h \neq 0$.
- Sign of $f^{(n)}(x_0)$ determines behavior:
  - $f^{(n)}(x_0) < 0$: $f(x_0 + h) - f(x_0) < 0 \Rightarrow$ **Local maximum.**
  - $f^{(n)}(x_0) > 0$: $f(x_0 + h) - f(x_0) > 0 \Rightarrow$ **Local minimum.**

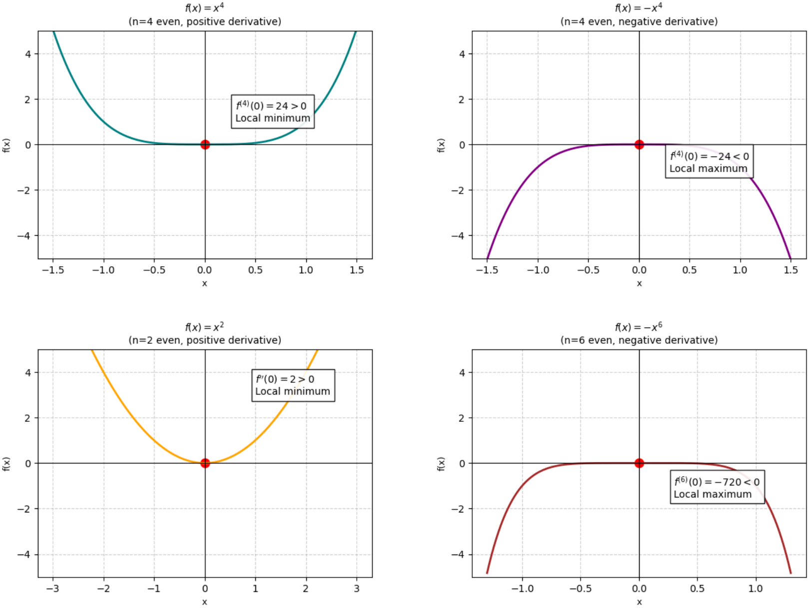
Example:

Let $f(x) = x^4$. At $x_0 = 0$:

- $f'(0) = f''(0) = f'''(0) = 0$, but $f^{(4)}(0) = 24 > 0$.
- $n = 4$ (even) $\Rightarrow$ Local minimum.


Case 1: n Even - Local Minimum Example
f(x) = x^4
Critical point (0,0)
Derivatives at 0:
f′(0) = 0
f″(0) = 0
f‴(0) = 0
f^(4)(0) = 24
Local minimum at x = 0
f^(4)(0) = 24 > 0
(Even n = 4)
16
14
12
10
8
6
4
2
0
f(x)
−2
−1
0
1
2
x

Case 1: Various Examples with Even n



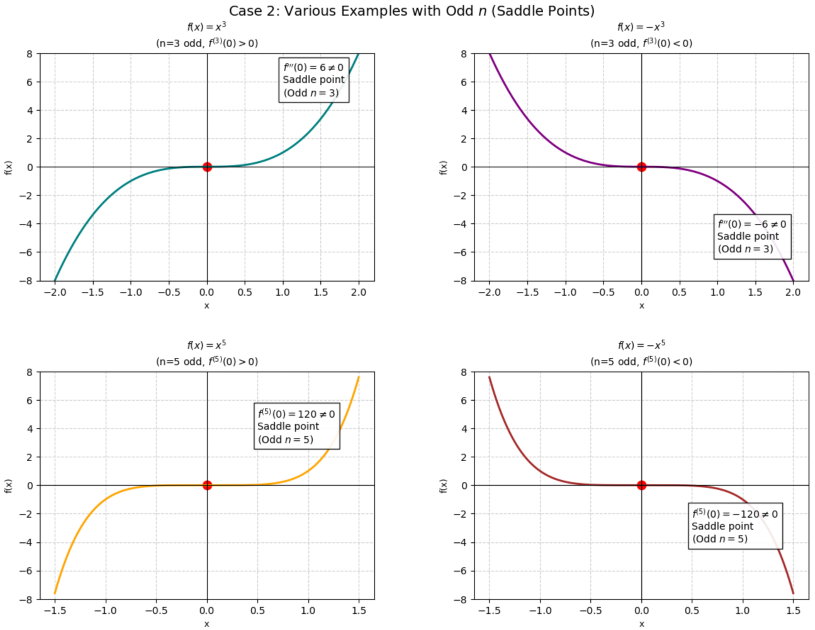
Case 2: n Odd

- If n is odd: h^n changes sign with h.
- For $h > 0$: $h^n > 0$.
- For $h < 0$: $h^n < 0$.
- Thus, $f(x_0 + h) - f(x_0)$ changes sign around x_0.
- $\Rightarrow$ **No local extremum** (saddle point).

example:

Let $f(x) = x^3$. At $x_0 = 0$:

- $f'(0) = f''(0) = 0$, but $f'''(0) = 6 \neq 0$.
- $n = 3$ (odd) $\Rightarrow$ No local optimum.


Case 2: Various Examples with Odd n (Saddle Points)
f(x) = x^3
(n=3 odd, f^(3)(0) > 0)
f'''(0) = 6 ≠ 0
Saddle point
(Odd n = 3)
f(x) = −x^3
(n=3 odd, f^(3)(0) < 0)
f'''(0) = −6 ≠ 0
Saddle point
(Odd n = 3)
f(x) = x^5
(n=5 odd, f^(5)(0) > 0)
f^(5)(0) = 120 ≠ 0
Saddle point
(Odd n = 5)
f(x) = −x^5
(n=5 odd, f^(5)(0) < 0)
f^(5)(0) = −120 ≠ 0
Saddle point
(Odd n = 5)
f(x)
x

Key Takeaways

- Use **Taylor's formula** when lower-order derivatives vanish.
- Identify the first non-zero derivative $f^{(n)}(x_0)$.
- **Even** n:
  - Maximum if $f^{(n)}(x_0) < 0$.
  - Minimum if $f^{(n)}(x_0) > 0$.
- **Odd** n: Saddle point (no extremum).

Application

Resolve ambiguous cases where the second-derivative test fails.

Oblique Asymptotes and Asymptotic Analysis

What is an Asymptote?

- A line a function approaches as $x \to \pm\infty$ **or** near discontinuities
- Three main types:
  1. Horizontal ($y = c$)
  2. Vertical ($x = a$)
  3. Oblique ($y = mx + b$)
- Key distinction: Functions **can intersect** oblique asymptotes finitely many times

Formal Definition

An **oblique asymptote** for $f(x)$ is a linear function $g(x) = mx + b$ such that:

$$\lim_{x \to \pm\infty} [f(x) - g(x)] = 0$$

Methodology Using Asymptotic Expansions

4-Step Procedure

1. **Prepare the function**:
   - For rational functions: $f(x) = \frac{g(x)}{h(x)}$
   - For transcendental functions: Isolate dominant terms
2. **Substitute $t = 1/x$**:

$$f(x) = f\left(\frac{1}{t}\right) \quad (t \to 0)$$

3. **Expand and truncate**:
   - Keep terms up to $\mathcal{O}(1)$
   - Discard $\mathcal{O}(t^2)$ and higher
4. **Reconstruct linear approximation**:

$$y = mx + b$$

Example 1: Oblique Asymptote for Exponential Function

Find the oblique asymptote of $f(x) = xe^{\frac{2}{x}} + 1$ as $x \to +\infty$.

Solution:

1. Taylor series expansion of e^t at $t = 0$:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

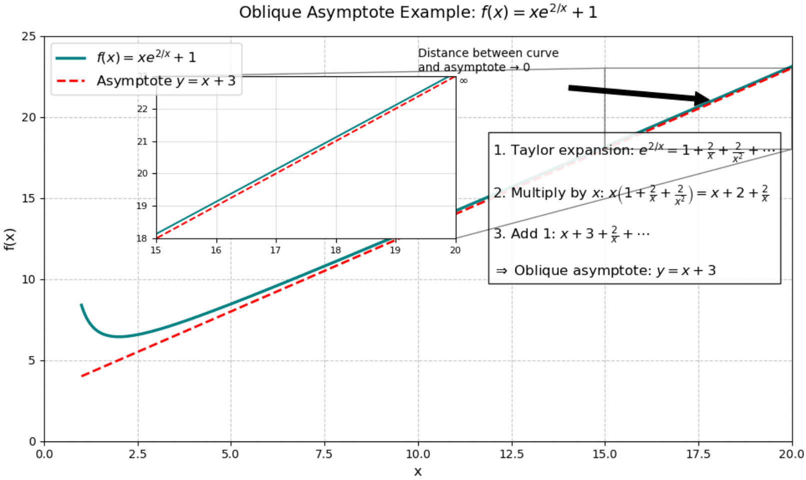
2. Substitute $t = \frac{2}{x}$:

$$e^{\frac{2}{x}} = 1 + \frac{2}{x} + \frac{4}{2x^2} + \frac{8}{6x^3} + \cdots$$

3. Multiply by x and add 1:

$$\begin{aligned} xe^{\frac{2}{x}} + 1 &= x\left(1 + \frac{2}{x} + \frac{2}{x^2} + \cdots\right) + 1 \\ &= x + 2 + \frac{2}{x} + \cdots + 1 \\ &= x + 3 + o\left(\frac{1}{x}\right) \end{aligned}$$

Conclusion: $y = x + 3$


Oblique Asymptote Example: $f(x) = xe^{2/x} + 1$
$f(x) = xe^{2/x} + 1$
Asymptote $y = x + 3$
Distance between curve
and asymptote → 0
∞
1. Taylor expansion: $e^{2/x} = 1 + \frac{2}{x} + \frac{2}{x^2} + \cdots$
2. Multiply by x: $x\left(1 + \frac{2}{x} + \frac{2}{x^2}\right) = x + 2 + \frac{2}{x}$
3. Add 1: $x + 3 + \frac{2}{x} + \cdots$
⇒ Oblique asymptote: $y = x + 3$
f(x)
x

Example 2: Oblique Asymptote for Rational Function

Find the oblique asymptote of $f(x) = \frac{x^3}{(x+1)^2}$ as $x \to +\infty$.

Solution:

1. Substitute $x = \frac{1}{t}$ $(t \to 0)$:

$$f\left(\frac{1}{t}\right) = \frac{1}{t^3} \cdot \frac{t^2}{(1+t)^2} = \frac{1}{t(1+t)^2}$$

2. Binomial expansion for $(1+t)^{-2}$:

$$(1+t)^{-2} = 1 - 2t + 3t^2 - 4t^3 + \cdots$$

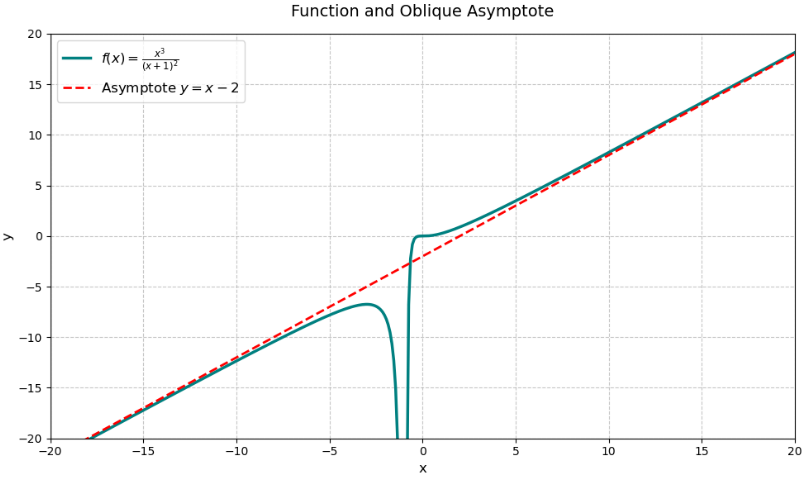
3. Multiply by $\frac{1}{t}$:

$$\frac{1}{t}(1 - 2t + 3t^2 - \cdots) = \frac{1}{t} - 2 + 3t - 4t^2 + \cdots$$

4. Substitute back $t = \frac{1}{x}$:

$$f(x) = x - 2 + \frac{3}{x} - \frac{4}{x^2} + \cdots$$

Conclusion: $\boxed{y = x - 2}$


Function and Oblique Asymptote
$f(x) = \frac{x^3}{(x+1)^2}$
Asymptote $y = x - 2$
y
x
20
15
10
5
0
−5
−10
−15
−20
−20
−15
−10
−5
0
5
10
15
20

5. Power Series & Taylor Series with Python SymPy

What is Python?

- A high-level, general-purpose programming language
- Key features:
  - Simple syntax (readability-focused)
  - Extensive ecosystem (NumPy, SciPy, Matplotlib, SymPy)
  - Free and open-source
- Widely used in scientific computing and education

What is SymPy?

- A **symbolic mathematics library** for Python
- Designed for:
  - Algebraic manipulations
  - Calculus (derivatives, integrals, limits, series)
  - Equation solving
  - Discrete math and logic
- Key advantages:
  - Exact arithmetic (no floating-point approximations)
  - Works with symbolic variables (e.g., x, y)

Why Use SymPy for Power Series?

- Automates tedious manual calculations
- Guarantees symbolic accuracy

Find the Maclaurin series expansion of $f(x) = \frac{1}{(1-x)(1+3x)}$.

Code:

```
from sympy import *
x = symbols('x')
f = 1/((1 - x)*(1 + 3*x))
taylor_series = series(f, x, 0, 5)  # Includes O(x^5)
print(taylor_series)
```

Output:

```
1 - 2*x + 7*x**2 - 20*x**3 + 61*x**4 + O(x**5)
```

Example 2: Exponential Function

Find the Maclaurin series expansion of $f(x) = e^x$.

Code:

```
from sympy import *
x = symbols('x')
f_exp = exp(x)
taylor_exp = series(f_exp, x, 0, 6)  # Up to x^5 term
print("\nTaylor series for e^x:")
pprint(taylor_exp)
```

Output:

```
 Taylor series for e^x:
          2     3     4     5        ⎛ 6⎞
         x     x     x     x         ⎜  ⎟
 1 + x + ── + ── + ── + ─── + O⎝x ⎠
         2     6     24    120
```

Example 3: Sine Function

Find the Maclaurin series expansion of $f(x) = \sin(x)$.

Code:

```
from sympy import *
x = symbols('x')
f = sin(x)
taylor= series(f, x, 0, 6)  # Up to x^5 term
pprint(taylor)
```

Output:

```
      3     5
     x     x      ⎛ 6⎞
x - ── + ─── + O⎝x ⎠
     6    120
```

Example 4: Natural Logarithm

Find the Maclaurin series expansion of $f(x) = \ln(1 + x)$.

Code:

```
from sympy import *
x = symbols('x')
f = log(1+x)
taylor= series(f, x, 0, 6)  # Up to x^5 term
pprint(taylor)
```

Output:

```
         2     3     4     5        
        x     x     x     x      ⎛ 6⎞
x  -  ─── + ─── - ─── + ─── + O⎝x ⎠
        2     3     4     5        
```

Example 5: Custom Function

Find the Maclaurin series expansion of $f(x) = \cos(x)\exp(x)$.

Code:

```
from sympy import *
x = symbols('x')
f_custom = cos(x) * exp(x)
taylor_custom = series(f_custom, x, 0, 5)  # Up to x^4 term
print("\nTaylor series for cos(x)*e^x:")
pprint(taylor_custom)
```

Output:

```
Taylor series for cos(x)*e^x:
          3    4
         x    x      ⎛ 5⎞
1 + x - ── - ── + O⎝x ⎠
         3    6
```

Thanks