Analysis I: Solutions of Tutorial Exercise Sheet 3

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Exercise 1: Domains of Definition

Determine the domain of definition for each function.

(1) $f(x) = \dfrac{x+1}{1-e^{1/x}}$

$$D_f =]-\infty, 0[\cup]0, +\infty[.$$

Explanation: The function is defined when $x \neq 0$ (to ensure $e^{1/x}$ is defined and the denominator is nonzero).

(2) $f(x) = \dfrac{1}{\sqrt{\sin x}}$

$$D_f = \bigcup_{k\in\mathbb{Z}}]2k\pi, \pi + 2k\pi[.$$

Explanation: The square root requires $\sin x \geq 0$, and since it is in the denominator, we need $\sin x > 0$.

(3) $f(x) = e^{\frac{1}{1-x}} \sqrt{x^2 - 1}$

$$D_f =]-\infty, -1] \cup]1, +\infty[.$$

Explanation: The exponential requires $x \neq 1$, and the square root requires $x^2 - 1 \geq 0$.

(4) $f(x) = (1 + \ln x)^{1/x}$

Writing $f(x) = e^{\frac{1}{x}\ln(1+\ln x)}$, then

$$D_f = \{x \in \mathbb{R} \mid x > 0 \land 1 + \ln x > 0\}$$

hence

$$D_f =]e^{-1}, +\infty[.$$

Explanation: We need $x > 0$ for $\ln x$ and $1 + \ln x > 0$ for the logarithm inside the exponent.

(5) $f(x) = \dfrac{1}{\lfloor x \rfloor}$

$$D_f =]-\infty, 0[\cup [1, +\infty[,$$

because $\lfloor x \rfloor = 0 \Leftrightarrow x \in [0, 1[$.

Explanation: The floor function is zero when $0 \leq x < 1$, so we exclude that interval.

(6) $f(x) = \begin{cases} \sqrt{x-2}, & x > 1, \\ \ln(x+2), & x \leq 1. \end{cases}$

$$D_f =]-2, 1] \cup [2, +\infty[.$$

Explanation: For $x > 1$, $\sqrt{x-2}$ requires $x \geq 2$; for $x \leq 1$, $\ln(x+2)$ requires $x > -2$.

Exercise 2: Limits

Calculate the following limits.

(1) $l_1 = \lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right)$

Solution. *Since* $|\sin(1/x)| \le 1$ *for all* $x \neq 0$*, we have*

$$-|x| \le x\sin(1/x) \le |x|.$$

As $x \to 0$*, both* $|x|$ *and* $-|x|$ *tend to* 0*. By the squeeze theorem,*

$$l_1 = 0.$$

(2) $l_2 = \lim_{x\to+\infty} x \sin\left(\frac{1}{x}\right)$

Solution. *Let* $t = 1/x$*. As* $x \to +\infty$*,* $t \to 0^+$*. Then*

$$l_2 = \lim_{t\to0^+} \frac{\sin t}{t} = 1.$$

(3) $l_3 = \lim_{x\to0} \frac{x - \sin(2x)}{x + \sin(3x)}$

Solution. *Divide numerator and denominator by* x *(for* $x \neq 0$*):*

$$l_3 = \lim_{x\to0} \frac{1 - \frac{\sin(2x)}{x}}{1 + \frac{\sin(3x)}{x}}.$$

Using $\lim_{u\to0} \frac{\sin u}{u} = 1$*, we get*

$$\frac{\sin(2x)}{x} = 2 \cdot \frac{\sin(2x)}{2x} \to 2, \quad \frac{\sin(3x)}{x} = 3 \cdot \frac{\sin(3x)}{3x} \to 3.$$

Thus,

$$l_3 = \frac{1-2}{1+3} = -\frac{1}{4}.$$

(4) $l_4 = \lim_{x\to0} \frac{\tan x}{x}$

Solution. *Write* $\tan x = \sin x/\cos x$*. Then*

$$l_4 = \lim_{x\to0} \frac{\sin x}{x\cos x} = \left(\lim_{x\to0} \frac{\sin x}{x}\right)\left(\lim_{x\to0} \frac{1}{\cos x}\right) = 1 \cdot 1 = 1.$$

(5) $l_5 = \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \tan x}$

Solution. *Simplify the expression:*

$$\begin{aligned}\frac{\sin x - \cos x}{1 - \tan x} &= \frac{\sin x - \cos x}{1 - \frac{\sin x}{\cos x}} \\ &= \frac{\sin x - \cos x}{\frac{\cos x - \sin x}{\cos x}} \\ &= \frac{\sin x - \cos x}{-(\sin x - \cos x)/\cos x} \\ &= -\cos x.\end{aligned}$$

Since $\sin(\pi/4) = \cos(\pi/4)$*, we must take the limit:*

$$l_5 = \lim_{x\to\pi/4}(-\cos x) = -\cos(\pi/4) = -\frac{\sqrt{2}}{2}.$$

(6) $l_6 = \lim_{x\to 0} \frac{\sin(x\ln x)}{x^2}$

Solution.

$$l_6 = \lim_{x\to 0^+} \frac{\sin(x\ln x)}{x\ln x} \cdot \frac{x\ln x}{x^2} = 1 \cdot \lim_{x\to 0^+} \frac{\ln x}{x} = -\infty.$$

(7) $l_7 = \lim_{x\to a^+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$

Solution. *Factor the denominator:* $\sqrt{x^2-a^2} = \sqrt{(x-a)(x+a)}$. *Write the expression as:*

$$\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}} + \frac{\sqrt{x-a}}{\sqrt{x^2-a^2}}.$$

For the first term:

$$\begin{aligned}
\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}} &= \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}\sqrt{x+a}} \\
&= \frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x-a}\sqrt{x+a}(\sqrt{x}+\sqrt{a})} \\
&= \frac{x-a}{\sqrt{x-a}\sqrt{x+a}(\sqrt{x}+\sqrt{a})} \\
&= \frac{\sqrt{x-a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})}.
\end{aligned}$$

For the second term:

$$\frac{\sqrt{x-a}}{\sqrt{x^2-a^2}} = \frac{\sqrt{x-a}}{\sqrt{x-a}\sqrt{x+a}} = \frac{1}{\sqrt{x+a}}.$$

Hence,

$$l_7 = \lim_{x\to a^+} \left(\frac{\sqrt{x-a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})} + \frac{1}{\sqrt{x+a}} \right) = 0 + \frac{1}{\sqrt{2a}} = \frac{1}{\sqrt{2a}}.$$

(8) $l_8 = \lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x$

Solution.

$$l_8 = \lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to+\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

and by making the change of variables: $t = \frac{1}{x}$, *we have*

$$l_8 = \lim_{t\to 0} e^{\frac{\ln(1+t)}{t}} = e.$$

(9) $l_9 = \lim_{x\to+\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right)$

Solution. *Use the identity* $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$*:*

$$\sin\sqrt{x+1} - \sin\sqrt{x} = 2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right).$$

Now,

$$\sqrt{x+1}-\sqrt{x} = \frac{1}{\sqrt{x+1}+\sqrt{x}} \to 0 \quad \text{as } x\to+\infty.$$

Hence $\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) \to 0$, *while the cosine term is bounded. Therefore,*

$$l_9 = 0.$$

(10) $l_{10} = \lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$

Solution. *Let* $t = x - 1$*, so* $x = t + 1$ *and* $t \to 0$*. Then*

$$1 - x = -t, \quad \frac{\pi x}{2} = \frac{\pi(t+1)}{2} = \frac{\pi t}{2} + \frac{\pi}{2}.$$

Using $\tan(\theta + \frac{\pi}{2}) = -\cot\theta = -\frac{1}{\tan\theta}$*, we get*

$$\begin{aligned} l_{10} &= \lim_{t \to 0} (-t) \tan\left(\frac{\pi t}{2} + \frac{\pi}{2}\right) \\ &= \lim_{t \to 0} (-t) \left(-\frac{1}{\tan(\pi t/2)}\right) \\ &= \lim_{t \to 0} \frac{t}{\tan(\pi t/2)}. \end{aligned}$$

Let $u = \pi t/2$*, so* $t = 2u/\pi$*. Then*

$$l_{10} = \lim_{u \to 0} \frac{2u/\pi}{\tan u} = \frac{2}{\pi} \lim_{u \to 0} \frac{u}{\tan u} = \frac{2}{\pi} \cdot 1 = \frac{2}{\pi}.$$

Exercise 3: Proofs With Definitions of Limits

Using the definition of the limit, show that:

(1) $\left(\lim_{x \to 4} (2x - 1) = 7\right) \Leftrightarrow (\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R}; |x - 4| < \alpha \Rightarrow |2x - 8| < \varepsilon)$

Solution.

$$|2x - 8| < \varepsilon \Leftrightarrow 2|x - 4| < \varepsilon \Leftrightarrow |x - 4| < \frac{\varepsilon}{2},$$

then it suffices to take $\alpha = \frac{\varepsilon}{2}$*.*

(2) $\left(\lim_{x \to +\infty} \frac{3x - 1}{2x + 1} = \frac{3}{2}\right) \Leftrightarrow \left(\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R}; x > \alpha \Rightarrow \left|\frac{3x - 1}{2x + 1} - \frac{3}{2}\right| < \varepsilon\right)$

Solution.

$$\left|\frac{3x - 1}{2x + 1} - \frac{3}{2}\right| < \varepsilon \Leftrightarrow \frac{5}{4x + 2} < \varepsilon \Leftrightarrow x > \frac{5 - 2\varepsilon}{4\varepsilon},$$

then it suffices to take $\alpha = \left|\frac{5 - 2\varepsilon}{4\varepsilon}\right|$*.*

(3) $\left(\lim_{x \to +\infty} \ln x = +\infty\right) \Leftrightarrow (\forall A > 0, \exists \alpha > 0, \forall x \in \mathbb{R}; x > \alpha \Rightarrow \ln x > A)$

Solution.

$$\ln x > A \Leftrightarrow x > e^A,$$

then it suffices to take $\alpha = e^A$*.*

(4) $\left(\lim_{x \to -3^+} \frac{4}{x + 3} = +\infty\right) \Leftrightarrow \left(\forall A > 0, \exists \alpha > 0, \forall x \in \mathbb{R}; -3 < x < -3 + \alpha \Rightarrow \frac{4}{x + 3} > A\right)$

Solution.

$$\frac{4}{x + 3} > A \Leftrightarrow x < \frac{4}{A} - 3,$$

then it suffices to take $\alpha = \frac{4}{A}$*.*

Exercise 4: Continuity

(1) Demonstrate that the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 3, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Solution. *For $x \neq 0$, we have $|\sin(1/x)| \leq 1$. Thus,*

$$|f(x)| = |x\sin(1/x)| \leq |x|.$$

As $x \to 0$, $|x| \to 0$, so by the squeeze theorem,

$$\lim_{x\to 0} f(x) = 0.$$

But $f(0) = 3$. Since $\lim_{x\to 0} f(x) \neq f(0)$, the function is discontinuous at $x = 0$.

(2) What is the redefinition of $f(0)$ that makes $f(x)$ continuous at $x = 0$?

Solution. *To make f continuous at $x = 0$, we need $f(0) = \lim_{x\to 0} f(x) = 0$. Thus, redefining $f(0) = 0$ makes the function continuous at $x = 0$. This type of discontinuity is called removable.*

Exercise 5: Continuity and Uniform Continuity

Demonstrate that the function $f(x) = x^2$ is:

(1) continuous at $x = 3$.

Solution. *We show that $\lim_{x\to 3} f(x) = f(3) = 9$. Using the ε-δ definition:*

For any $\varepsilon > 0$, we need to find $\delta > 0$ such that if $|x - 3| < \delta$, then $|x^2 - 9| < \varepsilon$.

Note that $|x^2 - 9| = |x - 3| \cdot |x + 3|$. If we restrict $\delta \leq 1$, then for $|x - 3| < 1$, we have $2 < x < 4$, so $|x + 3| < 7$. Thus,

$$|x^2 - 9| = |x - 3| \cdot |x + 3| < 7|x - 3|.$$

Choose $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$. Then if $|x - 3| < \delta$, we have

$$|x^2 - 9| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon.$$

Hence, f is continuous at $x = 3$.

(2) uniformly continuous on $]0, 1[$.

Solution. ***Method 1: Using the definition.***

We show that for any (given) real number $\varepsilon > 0$, we can find $\delta > 0$ such that for all $x, x_0 \in]0, 1[$, if $|x - x_0| < \delta$, then $|x^2 - x_0^2| < \varepsilon$, where δ depends only on ε and not on x_0.

Note that

$$|x^2 - x_0^2| = |x + x_0| \cdot |x - x_0|.$$

Since $x, x_0 \in]0, 1[$, we have $|x + x_0| < 2$. Therefore,

$$|x^2 - x_0^2| < 2|x - x_0|.$$

Choose $\delta = \frac{\varepsilon}{2}$. Then for any $x, x_0 \in]0, 1[$ with $|x - x_0| < \delta$, we have

$$|x^2 - x_0^2| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Since δ depends only on ε and not on x_0, f is uniformly continuous on $]0, 1[$.

Method 2: Using the theorem on closed intervals.

The function $f(x) = x^2$ is continuous on the closed interval $[0, 1]$. By the theorem that a continuous function on a closed interval is uniformly continuous, f is uniformly continuous on $[0, 1]$. Since $]0, 1[\subset [0, 1]$, it follows that f is uniformly continuous on $]0, 1[$ as well.

Exercise 6: Uniform Continuity of $f(x) = \frac{1}{x}$

Demonstrate that the function $f(x) = \frac{1}{x}$ is:

(1) not uniformly continuous on $]0, 1[$.

(2) uniformly continuous on $]2, +\infty[$.

(1)

Solution. ***Method 1: By definition (contradiction).***
Suppose, for contradiction, that f is uniformly continuous on $]0, 1[$. Then for $\varepsilon = 1$, there exists $\delta > 0$ such that for all $x, x_0 \in]0, 1[$, if $|x - x_0| < \delta$, then $\left|\frac{1}{x} - \frac{1}{x_0}\right| < 1$.

Choose $x = \frac{\delta}{2}$ and $x_0 = \frac{\delta}{4}$, provided $\delta < 1$ so that both points lie in $]0, 1[$. Then

$$|x - x_0| = \left|\frac{\delta}{2} - \frac{\delta}{4}\right| = \frac{\delta}{4} < \delta.$$

However,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{2}{\delta} - \frac{4}{\delta}\right| = \frac{2}{\delta}.$$

For any $\delta > 0$, we have $\frac{2}{\delta} > 1$ whenever $\delta < 2$. This contradicts the assumption that $\left|\frac{1}{x} - \frac{1}{x_0}\right| < 1$. Hence f is not uniformly continuous on $]0, 1[$.

Method 2: Direct estimate.
Let x_0 and $x_0 + \delta$ be any two points in $]0, 1[$. Then

$$|f(x_0) - f(x_0 + \delta)| = \left|\frac{1}{x_0} - \frac{1}{x_0 + \delta}\right| = \frac{\delta}{x_0(x_0 + \delta)}.$$

For any fixed $\delta > 0$, by choosing x_0 sufficiently close to 0, the expression $\frac{\delta}{x_0(x_0+\delta)}$ can be made larger than any positive number. Therefore, there exists no $\delta > 0$ that works uniformly for all $x_0 \in]0, 1[$. Thus, f is not uniformly continuous on $]0, 1[$.

Method 3: Using sequences.
Consider the sequences $x_n = \frac{1}{n}$ and $x_0^{(n)} = \frac{1}{n+1}$ for $n \in \mathbb{N}$. Both sequences are in $]0, 1[$. Then

$$|x_n - x_0^{(n)}| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \to 0 \quad \text{as } n \to \infty.$$

But

$$|f(x_n) - f(x_0^{(n)})| = |n - (n+1)| = 1.$$

Thus, for $\varepsilon = 1$, no matter how small $\delta > 0$, we can choose n sufficiently large so that $|x_n - x_0^{(n)}| < \delta$ while $|f(x_n) - f(x_0^{(n)})| = 1 \geq \varepsilon$. By the sequential criterion for uniform continuity, f is not uniformly continuous on $]0, 1[$.

(2)

Solution. *We show that f is uniformly continuous on $]2, +\infty[$. For any $x, x_0 \in]2, +\infty[$, we have*

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{x x_0}.$$

Since $x, x_0 > 2$, it follows that $x x_0 > 4$, and thus

$$|f(x) - f(x_0)| < \frac{|x - x_0|}{4}.$$

Given any $\varepsilon > 0$, choose $\delta = 4\varepsilon$. Then for all $x, x_0 \in]2, +\infty[$ with $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| < \frac{\delta}{4} = \varepsilon.$$

Since δ depends only on ε and not on the particular points x, x_0, the function f is uniformly continuous on $]2, +\infty[$.

Exercise 7: Differentiability Implies Continuity

Prove that if $f(x)$ has a derivative at $x = x_0$, then $f(x)$ must be continuous at x_0.

Solution. *Assume that f is differentiable at x_0. Then the derivative*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

To prove continuity at x_0, we need to show that

$$\lim_{h \to 0} f(x_0 + h) = f(x_0),$$

or equivalently,

$$\lim_{h \to 0} \big[f(x_0 + h) - f(x_0)\big] = 0.$$

For $h \neq 0$, we can write

$$f(x_0 + h) - f(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} \cdot h.$$

Taking the limit as $h \to 0$ and using the fact that the limit of a product is the product of the limits (when both limits exist), we obtain

$$\lim_{h \to 0} \big[f(x_0 + h) - f(x_0)\big] = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot \lim_{h \to 0} h.$$

The first limit is exactly $f'(x_0)$, and the second limit is 0. Hence,

$$\lim_{h \to 0} \big[f(x_0 + h) - f(x_0)\big] = f'(x_0) \cdot 0 = 0.$$

Therefore,

$$\lim_{h \to 0} f(x_0 + h) = f(x_0),$$

which means that f is continuous at x_0.

Exercise 8: Continuity and Differentiability of a Piecewise Function

Consider the function:

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(1) Study the continuity of $f(x)$ at $x = 0$.

Solution. *From Exercise 04 $f(x)$ is continuous at $x = 0$.*
To check continuity at $x = 0$, we verify whether

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

For $x \neq 0$, we have $|f(x)| = |x \sin(1/x)| \leq |x|$, since $|\sin(1/x)| \leq 1$. Hence,

$$-|x| \leq f(x) \leq |x|.$$

As $x \to 0$, both $-|x|$ and $|x|$ tend to 0. By the squeeze theorem,

$$\lim_{x \to 0} f(x) = 0.$$

Therefore, $\lim_{x \to 0} f(x) = f(0)$, and f is continuous at $x = 0$.

(2) Is the function $f(x)$ differentiable at $x = 0$?

Solution. *To check differentiability at $x = 0$, we examine the limit*

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}.$$

For $h \neq 0$, $f(h) = h \sin(1/h)$, so

$$\frac{f(h)}{h} = \sin\left(\frac{1}{h}\right).$$

Thus,

$$f'(0) = \lim_{h \to 0} \sin\left(\frac{1}{h}\right).$$

This limit does not exist because as $h \to 0$, $1/h \to \infty$, and $\sin(1/h)$ oscillates between -1 and 1 without approaching any single value. Hence, f is not differentiable at $x = 0$.

Exercise 9: Differentiability and Continuity of the Derivative

Consider the function:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(1) Is the function $f(x)$ differentiable at $x = 0$?

Solution. *To check differentiability at $x = 0$, we compute the limit of the difference quotient:*

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}.$$

For $h \neq 0$, $f(h) = h^2 \sin\left(\frac{1}{h}\right)$. Hence,

$$\frac{f(h)}{h} = h \sin\left(\frac{1}{h}\right).$$

Since $|\sin(1/h)| \leq 1$, we have $|h \sin(1/h)| \leq |h|$, so by the squeeze theorem,

$$\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

Therefore,

$$f'(0) = 0.$$

Thus, f is differentiable at $x = 0$ and its derivative there is 0.

(2) Study the continuity of $f'(x)$ at $x = 0$.

Solution. *For $x \neq 0$, we differentiate f using the product and chain rules:*

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x^2 \sin\left(\frac{1}{x}\right)\right) \\
&= x^2 \cdot \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) + \sin\left(\frac{1}{x}\right) \cdot \frac{d}{dx}(x^2) \\
&= x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + \sin\left(\frac{1}{x}\right) \cdot 2x \\
&= -\cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right).
\end{aligned}
$$

At $x = 0$, we already found $f'(0) = 0$. To check whether f' is continuous at 0, we examine

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left[-\cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right)\right].$$

As $x \to 0$, the term $2x\sin(1/x) \to 0$ (since $|2x\sin(1/x)| \leq 2|x|$), but $\cos(1/x)$ oscillates between -1 and 1 without approaching any limit. Hence,

$$\lim_{x\to 0} f'(x) \quad \textit{does not exist.}$$

Consequently, $f'(x)$ is not continuous at $x = 0$, despite the fact that $f'(0)$ exists.

Exercise 10: Differentiation

Differentiate the function f where $f(x)$ is:

1. $2x^{\frac{7}{2}}$
2. $x + \sqrt{x}$
3. $2ax^3 - \frac{x^2}{b} + c$
4. $\frac{x}{a} + \frac{b}{x} + \frac{x^2}{a^2} + \frac{a^2}{x^2}$
5. $\frac{nx^2}{x^{1/3}} + \frac{m}{x\sqrt{x}} + \frac{x^{1/3}}{\sqrt{x}}$
6. $\sin(\ln x)$
7. $\ln\left(\frac{1}{\cos x}\right)$
8. $\frac{\sinh^2 x}{e^x}$
9. $\frac{\cosh^2 x}{e^x}$
10. $\arctan x$
11. $\cos(\arcsin x)$
12. $\arctan\left(\frac{2x}{3+x}\right)$

(1)

$$f(x) = 2x^{\frac{7}{2}}.$$

Using the power rule $(x^\alpha)' = \alpha x^{\alpha-1}$:

$$f'(x) = 2 \cdot \frac{7}{2}x^{\frac{7}{2}-1} = 7x^{\frac{5}{2}}.$$

(2)

$$f(x) = x + \sqrt{x} = x + x^{1/2}.$$

$$f'(x) = 1 + \frac{1}{2}x^{-1/2} = 1 + \frac{1}{2\sqrt{x}}.$$

(3)

$$f(x) = 2ax^3 - \frac{x^2}{b} + c.$$

Treat a, b, c as constants:

$$f'(x) = 2a \cdot 3x^2 - \frac{2x}{b} = 6ax^2 - \frac{2x}{b}.$$

(4)

$$f(x) = \frac{x}{a} + \frac{b}{x} + \frac{x^2}{a^2} + \frac{a^2}{x^2}.$$

Rewrite using powers:

$$f(x) = \frac{1}{a}x + bx^{-1} + \frac{1}{a^2}x^2 + a^2x^{-2}.$$

$$f'(x) = \frac{1}{a} + b(-1)x^{-2} + \frac{1}{a^2} \cdot 2x + a^2(-2)x^{-3} = \frac{1}{a} - \frac{b}{x^2} + \frac{2x}{a^2} - \frac{2a^2}{x^3}.$$

(5)

$$f(x) = \frac{nx^2}{x^{1/3}} + \frac{m}{x\sqrt{x}} + \frac{x^{1/3}}{\sqrt{x}}.$$

Simplify each term:

$$\frac{nx^2}{x^{1/3}} = nx^{2-1/3} = nx^{5/3},$$

$$\frac{m}{x\sqrt{x}} = \frac{m}{x^{1+1/2}} = mx^{-3/2},$$

$$\frac{x^{1/3}}{\sqrt{x}} = x^{1/3-1/2} = x^{-1/6}.$$

So,

$$f(x) = nx^{5/3} + mx^{-3/2} + x^{-1/6}.$$

$$f'(x) = n \cdot \frac{5}{3}x^{5/3-1} + m \cdot \left(-\frac{3}{2}\right) x^{-3/2-1} - \frac{1}{6}x^{-1/6-1} = \frac{5n}{3}x^{2/3} - \frac{3m}{2}x^{-5/2} - \frac{1}{6}x^{-7/6}.$$

Alternatively,

$$f'(x) = \frac{5n}{3}x^{2/3} - \frac{3m}{2x^{5/2}} - \frac{1}{6x^{7/6}}.$$

(6)

$$f(x) = \sin(\ln x).$$

Using the chain rule:

$$f'(x) = \cos(\ln x) \cdot \frac{1}{x} = \frac{\cos(\ln x)}{x}.$$

(7)

$$f(x) = \ln\left(\frac{1}{\cos x}\right).$$

Simplify: $\ln(1/\cos x) = -\ln(\cos x)$. Then

$$f'(x) = -\frac{1}{\cos x} \cdot (-\sin x) = \frac{\sin x}{\cos x} = \tan x.$$

(8)

$$f(x) = \frac{\sinh^2 x}{e^x}.$$

Using the quotient rule or rewrite:

$$\sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}, \quad \text{so} \quad f(x) = \frac{e^{2x} - 2 + e^{-2x}}{4e^x} = \frac{1}{4}(e^x - 2e^{-x} + e^{-3x}).$$

Then

$$f'(x) = \frac{1}{4}(e^x + 2e^{-x} - 3e^{-3x}).$$

(9)

$$f(x) = \frac{\cosh^2 x}{e^x}.$$

Similarly,

$$\cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4}, \quad \text{so} \quad f(x) = \frac{1}{4}(e^x + 2e^{-x} + e^{-3x}).$$

Then

$$f'(x) = \frac{1}{4}(e^x - 2e^{-x} - 3e^{-3x}).$$

(10)

$$f(x) = \arctan x.$$

$$f'(x) = \frac{1}{1+x^2}.$$

(11)

$$f(x) = \cos(\arcsin x).$$

Let $u = \arcsin x$, then $\cos u = \sqrt{1-x^2}$ (for $x \in (-1, 1)$), so $f(x) = \sqrt{1-x^2}$.

$$f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

(12)

$$f(x) = \arctan\left(\frac{2x}{3+x}\right).$$

Let $u = \frac{2x}{3+x}$. Then $f'(x) = \frac{1}{1+u^2} \cdot u'$. Compute u' using the quotient rule:

$$u' = \frac{2(3+x) - 2x \cdot 1}{(3+x)^2} = \frac{6}{(3+x)^2}.$$

Also,

$$1 + u^2 = 1 + \left(\frac{2x}{3+x}\right)^2 = \frac{(3+x)^2 + 4x^2}{(3+x)^2} = \frac{9 + 6x + 5x^2}{(3+x)^2}.$$

Therefore,

$$f'(x) = \frac{1}{\frac{9+6x+5x^2}{(3+x)^2}} \cdot \frac{6}{(3+x)^2} = \frac{6}{9 + 6x + 5x^2}.$$

So,

$$f'(x) = \frac{6}{5x^2 + 6x + 9}.$$