Chapter 03: Functions of a Real Variable

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Definitions

Definition:

- A real function of a real variable is a mapping f from a set $E \subset \mathbb{R}$ to a set $F \subset \mathbb{R}$, written as:
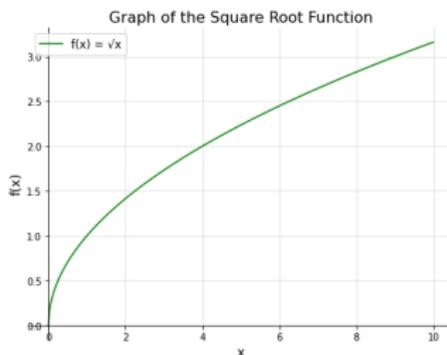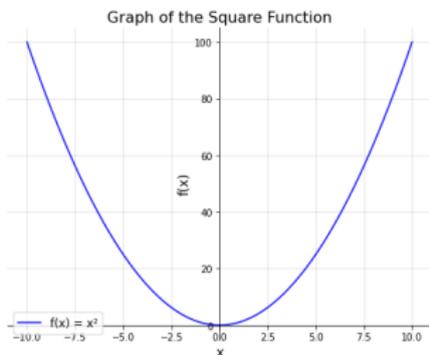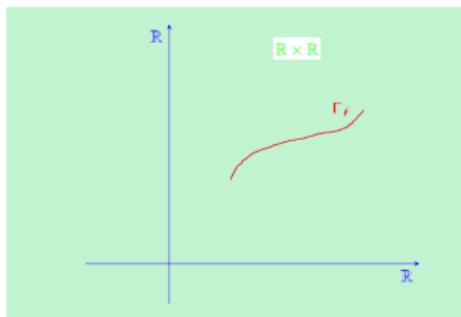
$$f : E \to F, \quad x \mapsto f(x).$$

- Here, x is called the real variable, and $f(x)$ is called the image of x under f.
- The domain of definition of f is the set of values $x \in E$ for which $f(x) \in F$, denoted D_f.
- The set of all functions from E to F is denoted as $F(E, F)$.

Set E

Rule (Map)
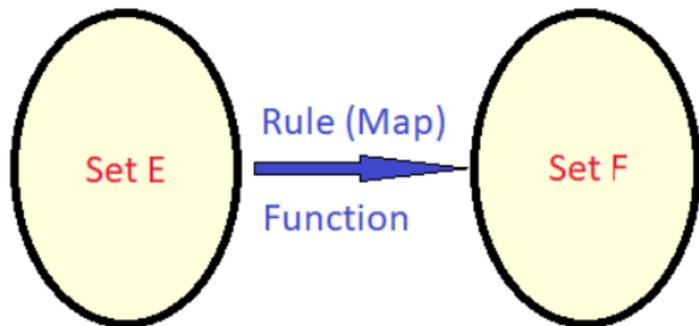
Function

Set F

The graph of f is the subset Γ_f of the Cartesian product $\mathbb{R} \times \mathbb{R}$ defined as:

$$\Gamma_f = \{(x, f(x)) \mid x \in E\}.$$

Graph of the Square Function

Graph of the Square Root Function

Definition: (Parity of a Function)
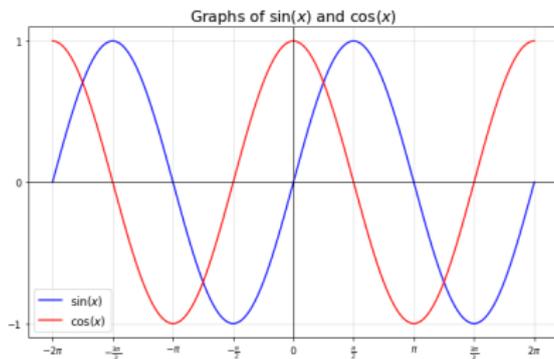
Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.

- f is called even if $\forall x \in D_f,\ f(-x) = f(x)$, meaning the graph of f is symmetric with respect to the y-axis.

- f is called odd if $\forall x \in D_f,\ f(-x) = -f(x)$, meaning the graph of f is symmetric with respect to the origin.

Definition: (Periodicity of a Function) A function f is said to be periodic if there exists a strictly positive real number T such that:
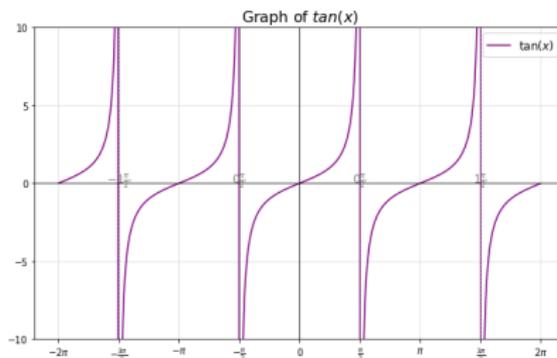
$$\forall x \in D_f,\ f(x + T) = f(x).$$

Examples:

- For $f(x) = \sin x$ or $f(x) = \cos x$, the period is $T = 2\pi$.

Graphs of sin(x) and cos(x)

- For $f(x) = \tan x$, the period is $T = \pi$.

Graph of $tan(x)$

- For $f(x) = x - \lfloor x \rfloor$, the period is $T = 1$.

Figure: Source: https://www.geeksforgeeks.org/fractional-part-function/
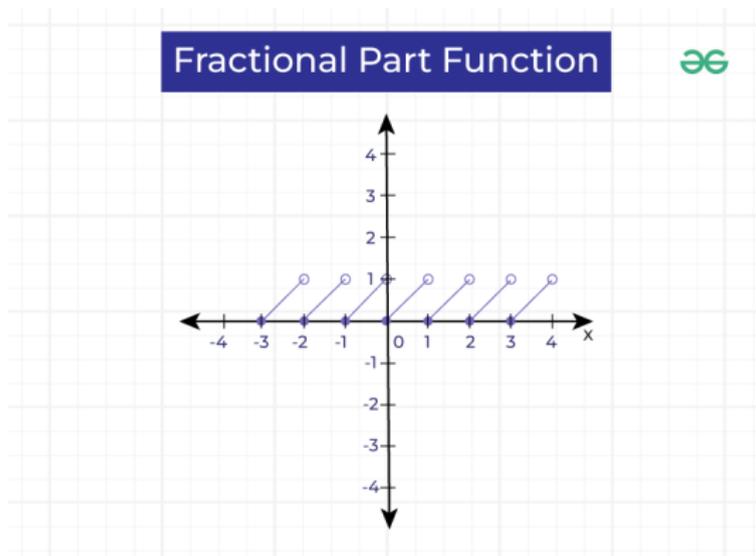
- For $f(x) = \cos\left(\frac{5x}{2}\right)$, the period is $T = \frac{4\pi}{5}$.

Remarks:
1. If f is even or odd, it suffices to study it over half its domain.
2. There exist functions that are neither even nor odd.
3. If f is periodic with period T, it suffices to study it over a single period.

Bounded Functions

1. A function $f(x)$ is said to be **bounded above** in an interval (or set) if there exists a constant M such that:

$$f(x) \leq M \quad \text{for all } x \text{ in the interval.}$$

Here, M is called an **upper bound** for $f(x)$.

2. Similarly, $f(x)$ is **bounded below** if there exists a constant m such that:

$$f(x) \geq m \quad \text{for all } x \text{ in the interval.}$$

In this case, m is called a **lower bound**.

3. If both conditions are satisfied—i.e., there exist constants m and M such that:

$$m \leq f(x) \leq M \quad \text{for all } x \text{ in the interval,}$$

then $f(x)$ is called a **bounded function**. To denote that $f(x)$ is bounded, we can write:

$$|f(x)| \leq P, \quad P > 0.$$

Examples

1) For $f(x) = \cos(x)$ in the interval $-\infty < x < \infty$:

- The function is bounded since $-1 \leq f(x) \leq 1$ for all x.
- $M = 1$ is an **upper bound**, and $m = -1$ is a **lower bound**.

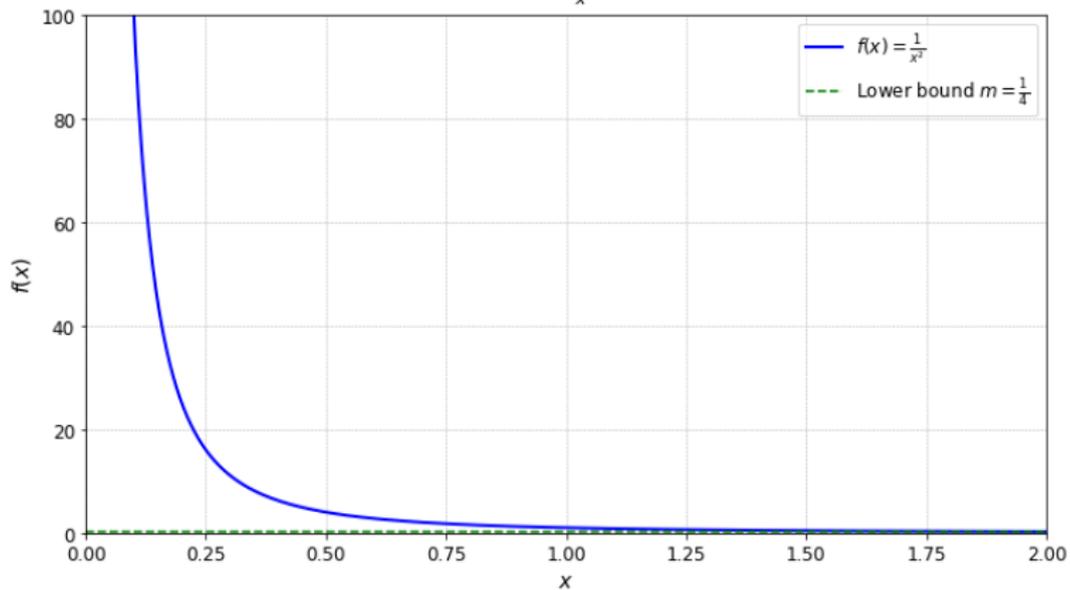2) For $f(x) = x^3$ in the interval $-2 \leq x \leq 2$:

- The function is bounded because $-8 \leq f(x) \leq 8$ in this interval.
- $M = 8$ is an **upper bound**, and $m = -8$ is a **lower bound**.

3) For $f(x) = \frac{1}{x^2}$ in the interval $0 < x \leq 2$:

- The function is **not bounded above**, as $f(x) \to \infty$ as $x \to 0^+$.
- However, it is bounded below with $f(x) \geq \frac{1}{4}$.

Graph of $f(x) = \frac{1}{x^2}$ in $0 < x \le 2$

Legend:
- $f(x) = \frac{1}{x^2}$
- Lower bound $m = \frac{1}{4}$

Definition of Monotonic Functions

Monotonically Increasing:

- A function $f(x)$ is monotonically increasing on an interval if, for any x_1, x_2 such that $x_1 < x_2$, we have:

$$f(x_1) \leq f(x_2).$$

- If $f(x_1) < f(x_2)$ strictly for all $x_1 < x_2$, then $f(x)$ is **strictly increasing**.

Monotonically Decreasing:

- A function $f(x)$ is monotonically decreasing on an interval if, for any x_1, x_2 such that $x_1 < x_2$, we have:

$$f(x_1) \geq f(x_2).$$

- If $f(x_1) > f(x_2)$ strictly for all $x_1 < x_2$, then $f(x)$ is **strictly decreasing**.

Examples:

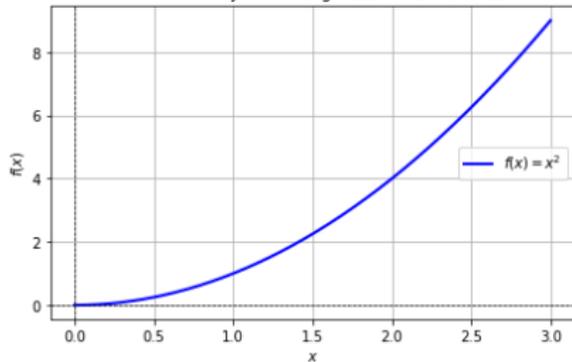**Monotonically Increasing:**

- The function $f(x) = x^2$ is monotonically increasing on $[0, \infty)$.
- The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$.
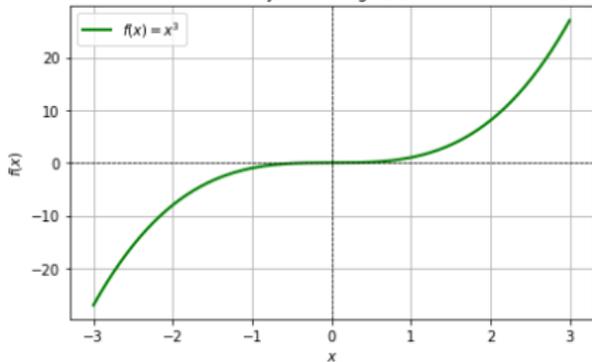
Monotonically Decreasing:

- The function $f(x) = -x$ is strictly decreasing on $\mathbb{R}$.

Monotonically Increasing: $f(x) = x^2$ (on $[0, \infty)$)

Strictly Increasing: $f(x) = x^3$

Strictly Decreasing: $f(x) = -x$
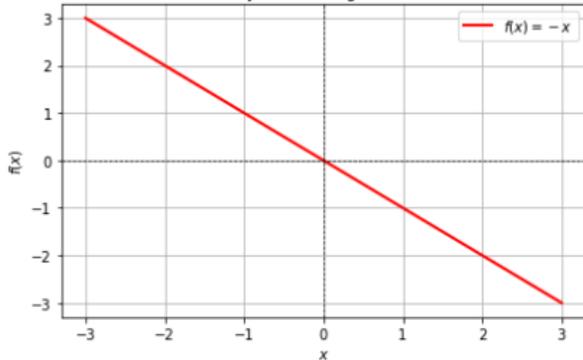
Introduction to Maxima and Minima

- The study of maxima and minima, or extreme values of functions, was a key motivation for the development of calculus in the 17th century.
- Extreme values are important in both mathematical theory and real-world applications.
- They are classified into:
  - **Relative extrema** (local maxima and minima).
  - **Absolute extrema** (global maxima and minima).

Relative Extrema

Definition:

- A function $f(x)$ has a **relative maximum** at c if there exists an interval (a, b) containing c such that:

$$f(x) < f(c) \quad \text{for all } x \neq c \text{ in } (a, b).$$

- Similarly, $f(x)$ has a **relative minimum** at c if:

$$f(x) > f(c) \quad \text{for all } x \neq c \text{ in } (a, b).$$

Key Points:

- Relative extrema are the "high points" (maximum) or "low points" (minimum) within a local neighborhood.
- Functions may have multiple relative extrema or none at all.

Absolute Extrema

Definition:

- A function $f(x)$ has an **absolute maximum** at c if:

$$f(x) \leq f(c) \quad \text{for all } x \text{ in the domain of } f.$$

- A function $f(x)$ has an **absolute minimum** at c if:

$$f(x) \geq f(c) \quad \text{for all } x \text{ in the domain of } f.$$

Key Points:

- Absolute extrema consider the entire domain of the function.
- Strictly increasing or decreasing functions may have absolute extrema at the endpoints of a closed interval.
- Absolute extrema are not always unique, as in the case of constant functions.

Examples and Observations

Examples:

- Relative extrema:
    - The highest point on a specific hill is a **relative maximum**.
    - The lowest point in a valley is a **relative minimum**.
- Absolute extrema:
    - The tallest hill overall is the **absolute maximum**.
    - The deepest valley is the **absolute minimum**.

Special Cases:

- Strictly increasing or decreasing functions have no relative extrema but may have absolute extrema on closed intervals.
- Constant functions have infinite absolute extrema, as all points are maxima and minima.
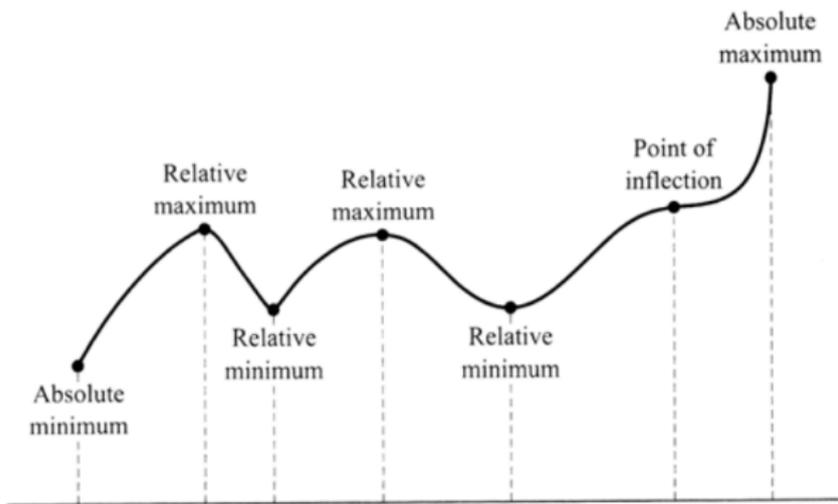
Figure 3.3

Figure: Source:Schaum's Outlines: Advanced Calculus By Robert C. Wrede, Murray R. Spiegel · 2011

Limits of a Function

Definition: Let f be a real-valued function defined on a set $E \subset \mathbb{R}$. We say that $f(x)$ tends toward a limit L as x approaches x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\forall x \in E, \ 0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

This is written as:

$$\lim_{x \to x_0} f(x) = L.$$

Theorems on limits:

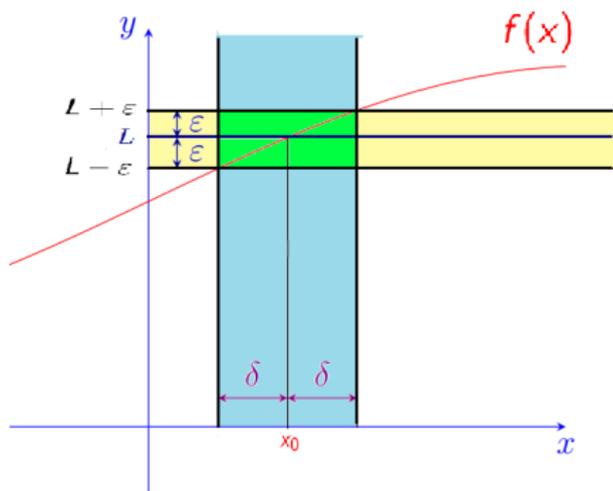1. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then:

$$\lim_{x \to x_0} [f(x) + g(x)] = L + M.$$

2. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then:

$$\lim_{x \to x_0} [f(x) \cdot g(x)] = L \cdot M.$$

3. If $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M$, and $M \neq 0$, then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Remark: If $f(x)$ is continuous at x_0, then:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Definition: A function $f(x)$ is said to have a limit L as x approaches infinity if, for every $\epsilon > 0$, there exists $N > 0$ such that:

$$\forall x > N, \ |f(x) - L| < \epsilon.$$

This is written as:

$$\lim_{x \to \infty} f(x) = L.$$

Theorem If f admits a limit at the point x_0, then this limit is unique.

Proof

Suppose, for contradiction, that f admits two different limits, l_1 and l_2 ($l_1 \neq l_2$). When x tends to x_0, we have:

$$\lim_{x \to x_0} f(x) = l_1 \iff$$

$$\forall \epsilon > 0, \exists \alpha_1 > 0, \forall x \in I, |x - x_0| < \alpha_1 \implies |f(x) - l_1| < \frac{\epsilon}{2}$$

$$\lim_{x \to x_0} f(x) = l_2 \iff$$

$$\forall \epsilon > 0, \exists \alpha_2 > 0, \forall x \in I, |x - x_0| < \alpha_2 \implies |f(x) - l_2| < \frac{\epsilon}{2}$$

Let $\epsilon > 0$, then:

$$|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)|$$

For $\alpha = \min(\alpha_1, \alpha_2)$, we have:

$$|l_1 - l_2| \leq |f(x) - l_1| + |f(x) - l_2| < \epsilon$$

Thus, for all $\epsilon > 0$ (no matter how small), it follows that $l_1 = l_2$.

Definition:

- f has a **left-hand limit** l_l as $x \to x_0^-$:

$$\lim_{x \to x_0^-} f(x) = l_l \iff$$

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I, \ x_0 - \alpha < x < x_0 \implies |f(x) - l_l| < \epsilon$$

- f has a **right-hand limit** l_r as $x \to x_0^+$:

$$\lim_{x \to x_0^+} f(x) = l_r \iff$$

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I, \ x_0 < x < x_0 + \alpha \implies |f(x) - l_r| < \epsilon$$

Remarks

1. If f has a limit l as $x \to x_0$, then:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = l$$

2. If f has a left-hand limit l_l and a right-hand limit l_r at x_0, and $l_l = l_r$, then:

$$\lim_{x \to x_0} f(x) = l_l = l_r$$

3. If $l_l \neq l_r$, then f does not have a limit as $x \to x_0$.

Infinity as a Limit

1.
$$\lim_{x \to x_0} f(x) = +\infty \iff$$

$$\forall A > 0, \exists \alpha > 0, \forall x \in I, |x - x_0| < \alpha \implies f(x) > A$$

2.
$$\lim_{x \to x_0} f(x) = -\infty \iff$$

$$\forall A < 0, \exists \alpha > 0, \forall x \in I, |x - x_0| < \alpha \implies f(x) < A$$

Limits at Infinity

①

$$\lim_{x \to +\infty} f(x) = l \iff \forall \epsilon > 0, \exists \alpha > 0, \forall x > \alpha \implies |f(x) - l| < \epsilon$$

②

$$\lim_{x \to -\infty} f(x) = l \iff \forall \epsilon > 0, \exists \alpha < 0, \forall x < \alpha \implies |f(x) - l| < \epsilon$$

Definition: A function $f(x)$ is said to be continuous at a point $x_0 \in D_f$ if:
$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $f(x)$ is continuous at every point in D_f, it is said to be continuous on D_f.

Continuity

A function $f(x)$ is said to be **continuous** at a point $x = x_0$ if the following conditions hold:

1. The limit $\lim_{x \to x_0} f(x)$ exists.
2. The value $f(x_0)$ exists (i.e., f is defined at x_0).
3. The limit matches the function's value:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

These conditions ensure the function behaves smoothly at x_0, with no jumps, gaps, or breaks.

Alternative Definition

A function $f(x)$ is continuous at x_0 if:

$\forall \epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

This means that we can make $f(x)$ arbitrarily close to $f(x_0)$ by taking x sufficiently close to x_0.

- If $f(x)$ is continuous at x_0, the graph of $f(x)$ near x_0 can be drawn without lifting the pencil from the paper.
- If there is a gap or jump in the graph at x_0, the function is **not continuous** there.

In simple terms, $f(x)$ is continuous at x_0 if:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Example 1: A Discontinuous Function

Let:
$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2, \\ 0 & \text{if } x = 2. \end{cases}$$

- For $x \neq 2$, $f(x) = x^2$, so $\lim_{x \to 2} f(x) = 4$.
- At $x = 2$, $f(2) = 0$.
- Since $\lim_{x \to 2} f(x) \neq f(2)$, $f(x)$ is **not continuous** at $x = 2$.

Example 2: A Continuous Function

Let $f(x) = x^2$ for all x:

- The limit $\lim_{x \to 2} f(x) = 4$.
- The value of the function at $x = 2$ is $f(2) = 4$.
- Since $\lim_{x \to 2} f(x) = f(2)$, $f(x)$ is **continuous** at $x = 2$.

Points of Discontinuity

Discontinuities are points where $f(x)$ fails to be continuous. These can occur if:

- $f(x)$ is undefined at the point (a gap),
- $f(x)$ jumps to a different value,
- The left-hand and right-hand limits do not match.

For example: - In Example 1, $f(x)$ is discontinuous at $x = 2$ because the limit and function value differ.

Property:

The sum, product, and quotient (where the denominator is nonzero) of continuous functions are continuous.

Examples:

1. The function $f(x) = x^2 + 3x + 2$ is continuous on $\mathbb{R}$.
2. The function $g(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

Theorem 1:

If $f(x)$ and $g(x)$ are continuous at $x = x_0$, then the following functions are also continuous at $x = x_0$:

$$f(x) + g(x), \quad f(x) - g(x), \quad f(x) \cdot g(x), \quad \frac{f(x)}{g(x)}$$

provided $g(x_0) \neq 0$.

Theorem 2:

The following functions are continuous over every finite interval:

- Polynomials.
- $\sin x$ and $\cos x$.
- Exponential functions a^x, where $a > 0$.

Theorem 3: If $f(x)$ is continuous at x_0, and $g(y)$ is continuous at $y_0 = f(x_0)$, then the composite function $g(f(x))$ is continuous at x_0.

Theorem 4: If $f(x)$ is continuous over a closed interval, then it is bounded on that interval.

Theorem 5: If $f(x)$ is continuous at x_0 and $f(x_0) > 0$ (or $f(x_0) < 0$), then there exists an interval around x_0 where $f(x) > 0$ (or $f(x) < 0$).

Theorem 6: If $f(x)$ is continuous and strictly monotonic (strictly increasing or decreasing) on an interval, then its inverse $f^{-1}(x)$ exists, is single-valued, and is also continuous.

Theorem 7: (Intermediate Value Theorem) If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) = A$, $f(b) = B$, then for any C between A and B, there exists at least one $c \in [a, b]$ such that:
$$f(c) = C.$$

Theorem 8: (Zero Value Theorem) If $f(x)$ is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists at least one $c \in (a, b)$ such that:

$$f(c) = 0.$$

Theorem 9: (Extreme Value Theorem) If $f(x)$ is continuous on a closed interval $[a, b]$, then it attains both its maximum value M and minimum value m at some points in the interval:

$$M = \max_{x \in [a,b]} f(x), \quad m = \min_{x \in [a,b]} f(x).$$

Theorem 10: If $f(x)$ is continuous on $[a, b]$, then its least upper bound (supremum) and greatest lower bound (infimum) are attained within the interval.

A function $f(x)$ is piecewise continuous on (a, b) if the interval can be divided into a finite number of subintervals where $f(x)$ is continuous, and $f(x)$ has finite left-hand and right-hand limits at any points of discontinuity.

Definition: A function $f(x)$ is uniformly continuous on an interval if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$|f(x_1) - f(x_2)| < \epsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta,$$

where x_1, x_2 are any two points in the interval.

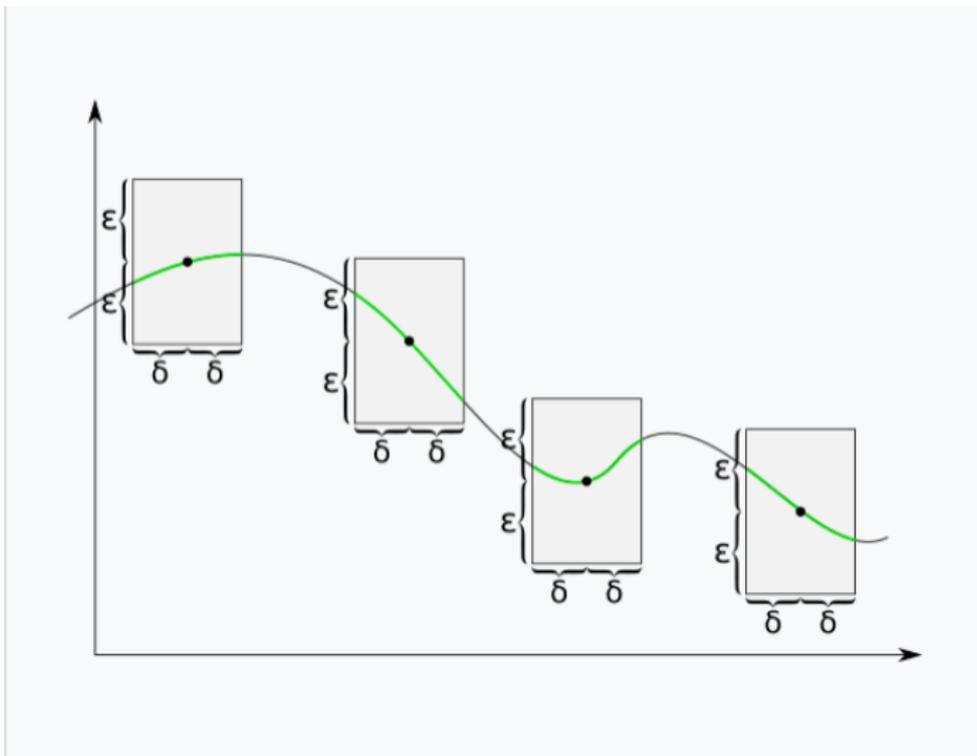**Theorem:** If $f(x)$ is continuous on a closed interval, it is also uniformly continuous on that interval.

Figure: Source:https://en.wikipedia.org

COUNTEREXAMPLE $f(x) = 1/x$ **NOT UNIFORMLY CONTINUOUS**

Figure:
Source:https://math.stackexchange.com/questions/2283008/uniform-continuity-of-function
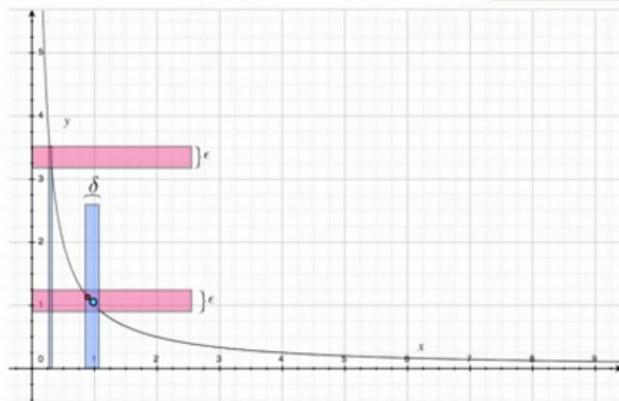
Derivatives of a Function

Definition: Let $f(x)$ be a real-valued function defined on an open interval containing x_0. The derivative of f at x_0, denoted $f'(x_0)$, is defined as:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. If $f'(x_0)$ exists, $f(x)$ is said to be differentiable at x_0.

Note: The derivative can be equivalently defined in several ways. Here are three common alternative formulations:

Example 1: Using $x \to x_0$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Example 2: Using the increment Δx

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Example 3: Using the left-hand difference quotient

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h},$$

provided this limit exists and equals the standard two-sided limit.

Higher-Order Derivatives: If $f'(x)$ is differentiable, the derivative of $f'(x)$ is called the second derivative of $f(x)$, denoted $f''(x)$. Similarly, higher-order derivatives are defined recursively.

Examples of Differentiation

1. For $f(x) = x^2$, the derivative is:

$$f'(x) = 2x.$$

2. For $g(x) = \sin x$, the derivative is:

$$g'(x) = \cos x.$$

3. For $h(x) = \ln x$, the derivative is:

$$h'(x) = \frac{1}{x}.$$

4. For $k(x) = e^x$, the derivative is:

$$k'(x) = e^x.$$

Right-Hand Derivative:

$$f'_+(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h},$$

where h approaches 0 from positive values.

Left-Hand Derivative:

$$f'_-(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h},$$

where h approaches 0 from negative values.

Condition for Differentiability:

$$f(x) \text{ is differentiable at } x_0 \iff f'_+(x_0) = f'_-(x_0).$$

Differentiability in an Interval

A function $f(x)$ is said to be **differentiable** in an interval if it has a derivative at every point in the interval.

Closed Interval Case: For $f(x)$ defined in $[a, b]$, $f(x)$ is differentiable if:

- $f'(x_0)$ exists for all x_0 where $a < x_0 < b$.
- The right-hand derivative exists at $x = a$ and the left-hand derivative exists at $x = b$.

Continuous Derivatives: If $f(x)$ has a continuous derivative, it is said to be **continuously differentiable**.

A function $f(x)$ is **sectionally differentiable** in $[a, b]$ if its derivative $f'(x)$ is piecewise continuous in the interval.
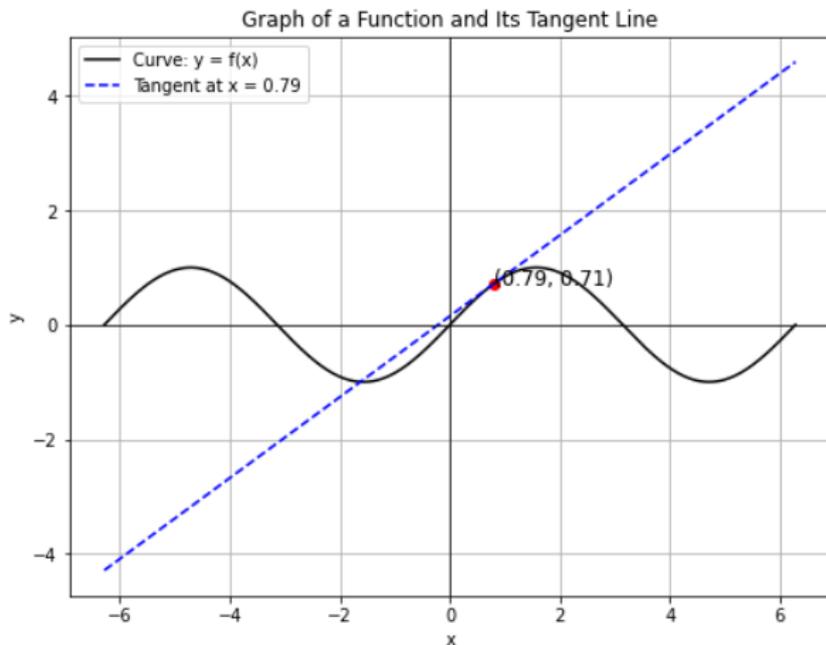
Example: Sectional continuity is commonly observed in functions defined piecewise over different subintervals of $[a, b]$.

Equation of the Tangent Line

The equation of the tangent line to the curve $y = f(x)$ at $x = x_0$ is:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This equation represents the linear approximation of $f(x)$ near x_0.

Graph of a Function and Its Tangent Line

Continuity and Differentiability

A function can be continuous but not differentiable at a point. This occurs at sharp corners or cusps in the graph. At such points, multiple tangent lines may exist. For instance, point P in the figure has two tangent lines with slopes $f_1'(x_0)$ and $f_2'(x_0)$.

Graph of a Continuous but Non-Differentiable Function at Point P
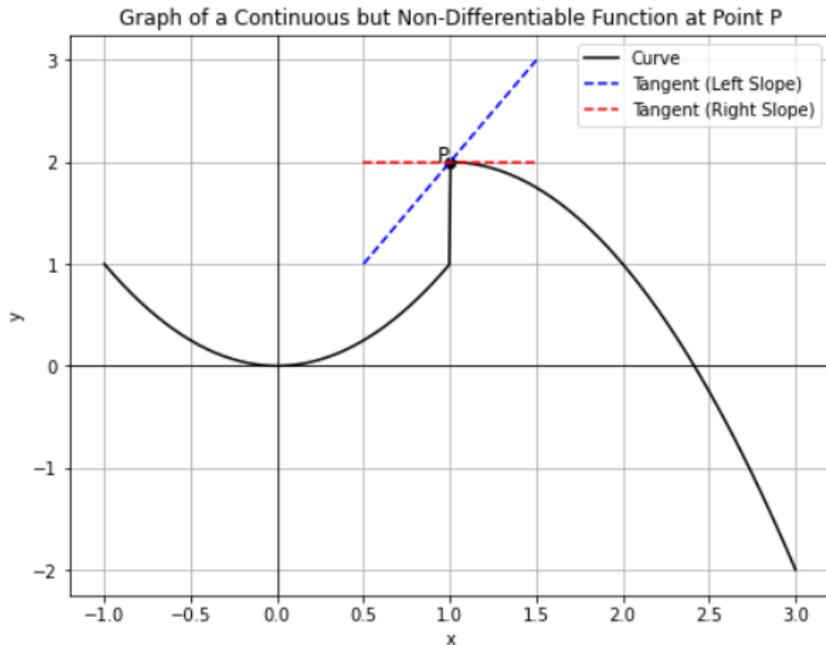
Differentials

Let $\Delta x = dx$ be a small increment in x. The corresponding change in $y = f(x)$ is:

$$\Delta y = f(x + \Delta x) - f(x).$$

For differentiable functions, this change can be approximated as:

$$\Delta y = f'(x)\Delta x + \varepsilon\Delta x,$$

where $\varepsilon \to 0$ as $\Delta x \to 0$. The term $dy = f'(x)dx$ is the differential of y, representing the principal part of Δy.

For differentiable functions $f(x)$ and $g(x)$, the following rules apply:

- **Sum Rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$,

- **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$,

- **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$,

- **Chain Rule:** If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Common functions

Theorem

A function f that is continuous and strictly monotonic from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$ is bijective on I to $f(I)$. Its inverse function $f^{-1} : f(I) \to I$ exists, is continuous, and follows the monotonicity of f.

Proof:

The function f is surjective from I to $f(I)$. Since f is strictly monotonic, it is also injective, making f bijective on $f(I)$. Thus, f^{-1} exists and follows the monotonicity of f. For example, if f is strictly increasing, let $y_1, y_2 \in f(I)$ such that $y_1 < y_2$. Then:

$$y_1 \neq y_2 \implies f^{-1}(y_1) \neq f^{-1}(y_2)$$

and there exist $x_1, x_2 \in I$ such that:

$$f^{-1}(y_1) = x_1, \quad f^{-1}(y_2) = x_2.$$

If we assume $x_1 > x_2$, then since f is strictly increasing, we get:

$$f(x_1) > f(x_2),$$

which contradicts $y_1 < y_2$. Hence:

$$x_1 < x_2 \iff f^{-1}(y_1) < f^{-1}(y_2).$$

Thus, f^{-1} is strictly increasing. Moreover, since f is continuous on I, $f(I)$ is an interval, and f^{-1} is continuous.

Definition: The exponential function is defined as $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The exponential function, denoted by exp, is the unique differentiable function on $\mathbb{R}$ that is equal to its derivative and satisfies $\exp(0) = 1$.

Properties:

1. $\exp(x) > 0, \forall x \in \mathbb{R}$.
2. $\exp(x + y) = \exp(x)\exp(y), \forall x, y \in \mathbb{R}$.
3. Using Euler's notation: $\exp(x) = e^x$, where $e \approx 2.718$.
4. The function exp is strictly increasing on $\mathbb{R}$.
5. $e^x = e^y \iff x = y, \quad e^x < e^y \iff x < y$.
6. exp is a bijection from $\mathbb{R}$ to $\mathbb{R}_*^+$.

Logarithmic Functions

Definition: The logarithmic function is given by $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$. This function is the inverse of the exponential function. When $a = e \approx 2.71828$, the function becomes $f(x) = \ln x$, called the *natural logarithm*.

The natural logarithm function is defined on $(0, +\infty)$ to $\mathbb{R}$ such that:

$$\forall x > 0 : x = e^y \iff y = \ln x.$$

Properties:

1. $\ln 1 = 0, \quad \ln e = 1$.
2. $\ln(e^x) = x, \quad e^{\ln x} = x, \quad \forall x > 0$.
3. $\ln$ is strictly increasing on $(0, +\infty)$.
4. $\ln(xy) = \ln x + \ln y, \quad \ln\left(\frac{1}{y}\right) = -\ln y$.
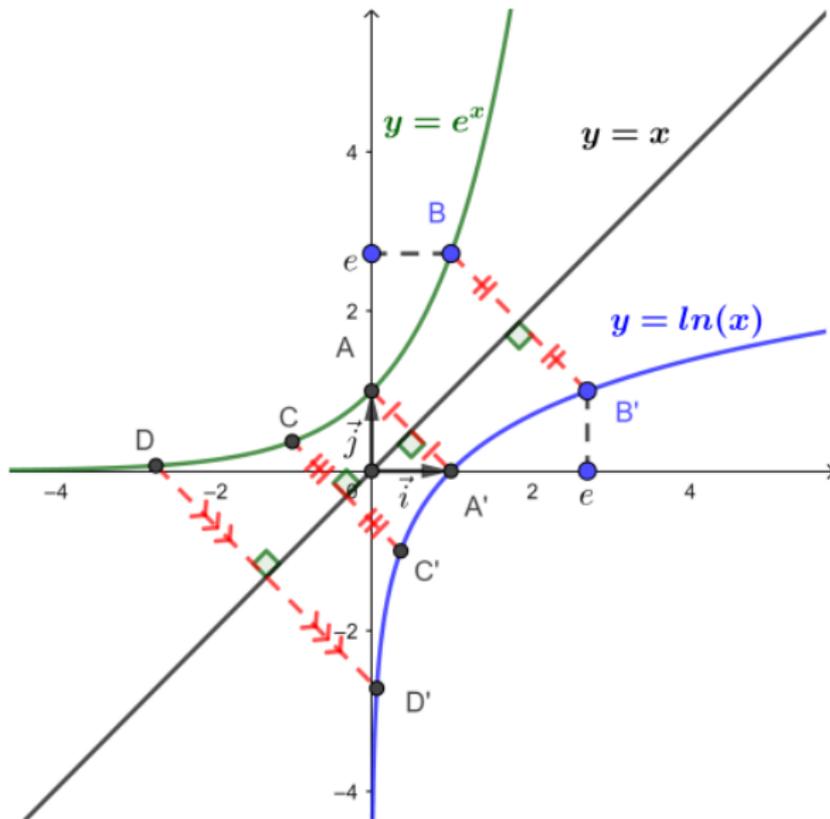5. $\ln(x^n) = n \ln x, \quad \forall n \in \mathbb{N}$.

Figure: Source: BOUHARIS Epouse, OUDJDI DAMERDJI Amel, Cours et exercices corrigés d'Analyse 1, Première année Licence ML

The standard trigonometric functions include:

$$\sin x, \quad \cos x, \quad \tan x = \frac{\sin x}{\cos x}, \quad \csc x = \frac{1}{\sin x},$$

$$\sec x = \frac{1}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}.$$

- The variable x is generally expressed in radians (π radians = $180°$). - For real values of x, $\sin x$ and $\cos x$ lie in the range $[-1, 1]$.

Key Properties of Trigonometric Functions

1. **Pythagorean Identities:**

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$$

2. **Angle Addition and Subtraction Formulas:**

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

3. **Sign Change Properties:**

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x.$$

Function arcsin

The function:

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1], \quad x \mapsto f(x) = \sin x$$

is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly increasing. We have:

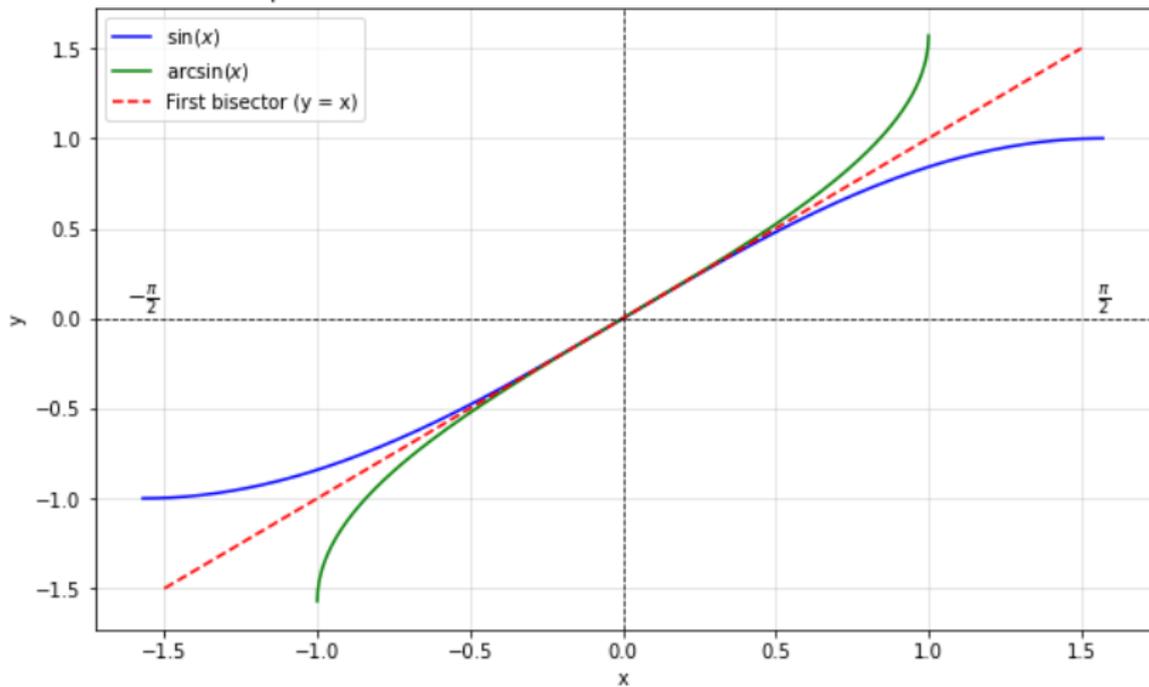$$f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1, 1]$$

and

$$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad y \mapsto f^{-1}(y) = \arcsin y.$$

Thus, we get:

$$\arcsin y = x \iff \sin x = y, \quad -1 \le y \le 1, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Graphs of $\sin(x)$, $\arcsin(x)$, and the First Bisector (Restricted Domain)

Inverse Trigonometric Functions: arccos

Function arccos

The function:

$$f : [0, \pi] \to [-1, 1], \quad x \mapsto f(x) = \cos x$$

is continuous and strictly decreasing on $[0, \pi]$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly decreasing. We have:
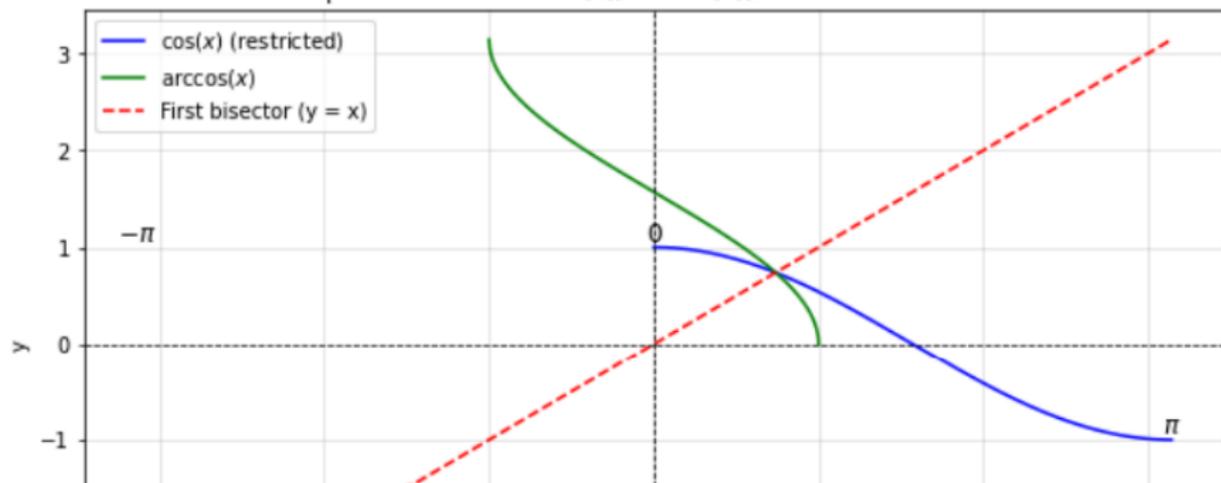
$$f([0, \pi]) = [-1, 1]$$

and

$$f^{-1} : [-1, 1] \to [0, \pi], \quad y \mapsto f^{-1}(y) = \arccos y.$$

Thus, we get:

$$\arccos y = x \iff \cos x = y, \quad -1 \le y \le 1, \quad 0 \le x \le \pi.$$

Graphs of Restricted cos(x), arccos(x), and the First Bisector

Inverse Trigonometric Functions: arctan

Function arctan

The function:

$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}, \quad x \mapsto f(x) = \tan x = \frac{\sin x}{\cos x}$$
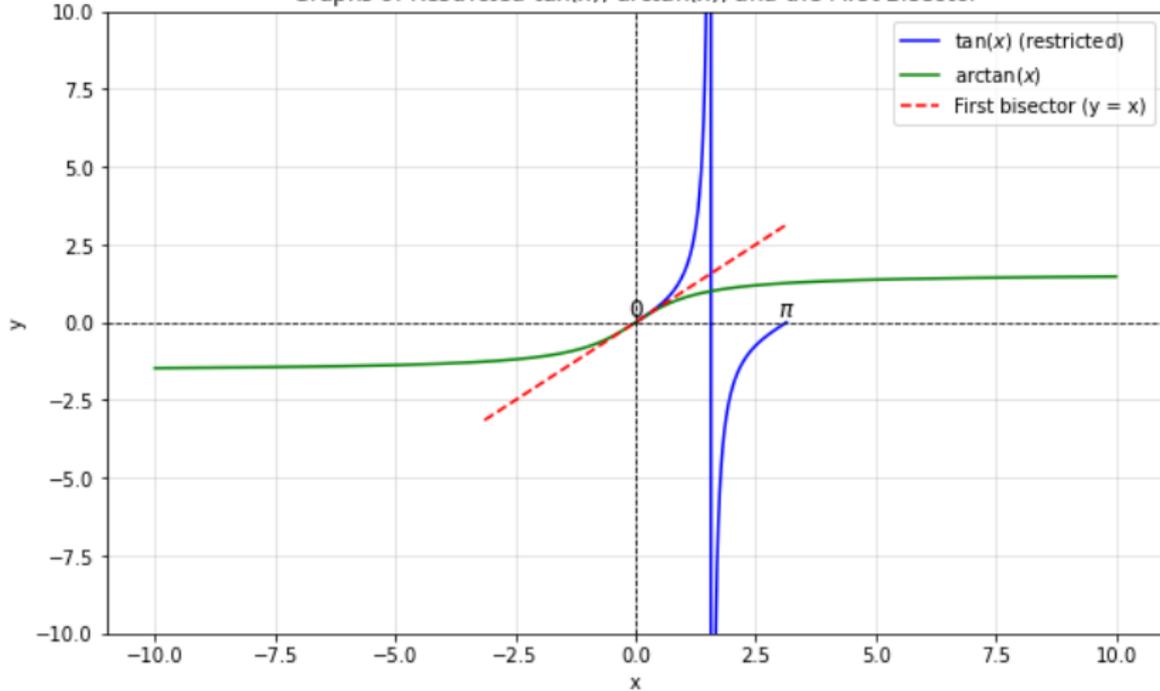
is continuous and strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly increasing. We have:

$$f\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) = \mathbb{R}$$

and

$$f^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y \mapsto f^{-1}(y) = \arctan y.$$

Graphs of Restricted tan(x), arctan(x), and the First Bisector

- — tan(x) (restricted)
- — arctan(x)
- --- First bisector (y = x)

Function arccot

The function:

$$f : (0, \pi) \to (-\infty, +\infty), \quad x \mapsto f(x) = \cot x = \frac{\cos x}{\sin x}$$

is continuous and strictly decreasing on $(0, \pi)$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly decreasing. We have:

$$f((0, \pi)) = (-\infty, +\infty)$$

and

$$f^{-1} : (-\infty, +\infty) \to (0, \pi), \quad y \mapsto f^{-1}(y) = \text{arccot } y.$$

Thus, we get:

$$\text{arccot } y = x \iff \cot x = y, \quad 0 < x < \pi.$$

Properties of Inverse Trigonometric Functions

Properties:

1. For all $x \in [-1, 1]$:

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

2. If $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

$$\sin t = x \iff \arcsin x = t.$$

Otherwise:

$$\sin t = x \iff t = \arcsin x + 2k\pi \text{ or } t = (\pi - \arcsin x) + 2k\pi, \quad k \in \mathbb{Z}.$$

3. If $t \in [0, \pi]$:

$$\cos t = x \iff \arccos x = t.$$

Otherwise:

$$\cos t = x \iff t = \arccos x + 2k\pi \text{ or } t = -\arccos x + 2k\pi, \quad k \in \mathbb{Z}.$$

Hyperbolic Functions

Definition: The hyperbolic cosine function, cosh, and hyperbolic sine function, sinh, are defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Properties:

1. $\cosh^2 x - \sinh^2 x = 1$.
2. $\frac{d}{dx} \cosh x = \sinh x, \quad \frac{d}{dx} \sinh x = \cosh x$.
3. $\cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x$.

Properties: $\cosh x$ is even, continuous, and strictly increasing on $[0, +\infty)$. Its inverse function $\cosh^{-1}$ exists, is continuous, and strictly increasing. We have:
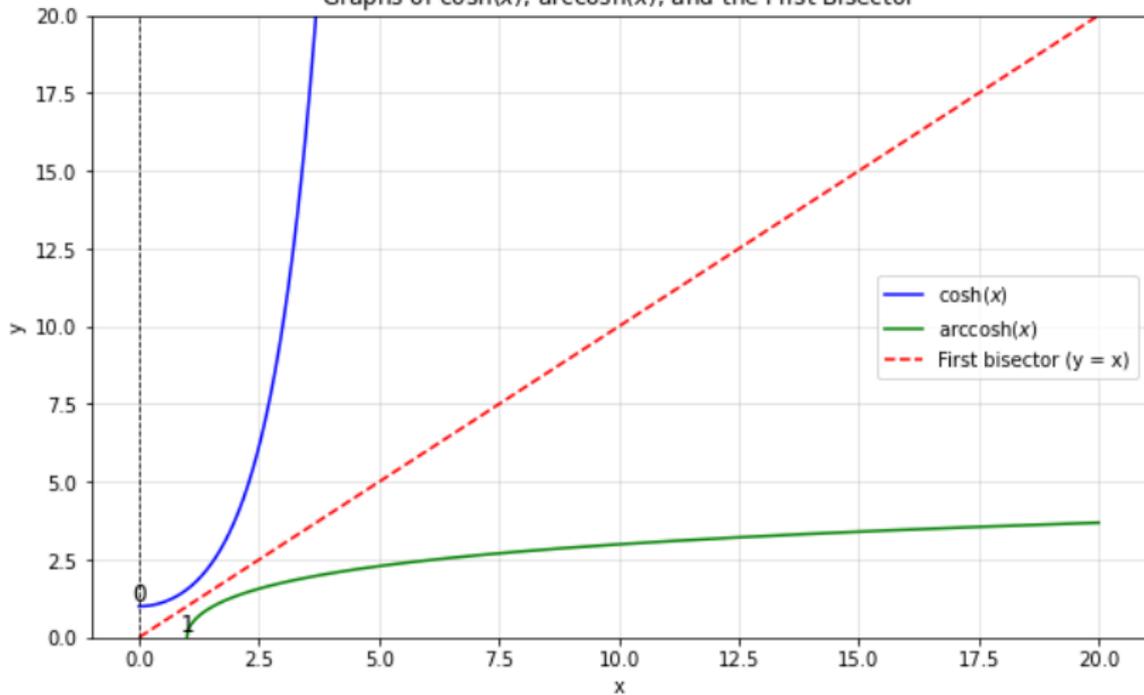
$$f([0, +\infty)) = [1, +\infty)$$

and

$$f^{-1} : [1, +\infty) \to [0, +\infty), \quad y \mapsto f^{-1}(y) = \text{arccosh}(y).$$

Thus, we get:

$$\text{arccosh}(y) = x \iff \cosh x = y, \quad x \geq 0.$$

Graphs of cosh(x), arccosh(x), and the First Bisector

— cosh(x)
— arccosh(x)
- - - First bisector (y = x)

Hyperbolic Sine Function

Definition: Hyperbolic Sine Function sinh:

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sinh x = \frac{e^x - e^{-x}}{2}$$

Properties: $\sinh x$ is odd, continuous, and strictly increasing on $\mathbb{R}$. Its inverse function $\sinh^{-1}$ exists, is continuous, and strictly increasing. We have:

$$f(\mathbb{R}) = \mathbb{R}$$

and

$$f^{-1} : \mathbb{R} \to \mathbb{R}, \quad y \mapsto f^{-1}(y) = \operatorname{arcsinh}(y).$$

Thus, we get:

$$\operatorname{arcsinh}(y) = x \iff \sinh x = y, \quad x \in \mathbb{R}.$$

Hyperbolic Tangent Function

Definition: Hyperbolic Tangent Function tanh:

$$f : \mathbb{R} \to (-1, 1), \quad x \mapsto \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Properties: $\tanh x$ is odd, continuous, and strictly increasing on $\mathbb{R}$. Its inverse function $\tanh^{-1}$ exists, is continuous, and strictly increasing. We have:

$$f(\mathbb{R}) = (-1, 1)$$

and

$$f^{-1} : (-1, 1) \to \mathbb{R}, \quad y \mapsto f^{-1}(y) = \operatorname{arctanh}(y).$$

Thus, we get:

$$\operatorname{arctanh}(y) = x \iff \tanh x = y, \quad |y| < 1.$$

Hyperbolic Cotangent Function

Definition: Hyperbolic Cotangent Function coth:

$$f : (0, +\infty) \to (1, +\infty), \quad x \mapsto \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Properties: $\coth x$ is odd, continuous, and strictly decreasing on $(0, +\infty)$. Its inverse function $\coth^{-1}$ exists, is continuous, and strictly decreasing. We have:

$$f((0, +\infty)) = (1, +\infty)$$

and

$$f^{-1} : (1, +\infty) \to (0, +\infty), \quad y \mapsto f^{-1}(y) = \mathsf{argcoth}(y).$$

Thus, we get:

$$\mathsf{argcoth}(y) = x \iff \coth x = y, \quad x > 0.$$

Properties

1. $\cosh x + \sinh x = e^x$:
2. $\cosh x - \sinh x = e^{-x}$:
3. $\cosh^2 x - \sinh^2 x = 1$:
4. $1 - \tanh^2 x = \operatorname{sech}^2 x$:
5. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$:
6. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$:

Expression in Logarithmic Form

The inverse functions of hyperbolic functions can be expressed using the natural logarithm as follows:

$$arctanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad x \in]-1,1[$$

$$arccothx = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad x \in]-1,-1[\cup[1,+1[$$

$$arcsinhx = \ln\left(x + \sqrt{1+x^2}\right), \quad x \in \mathbb{R}$$

$$arccoshx = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$$

Proof:

1.

- Let $x \in]-1, 1[$, and set $\text{arctanh} x = y$.

$$\tanh y = x \implies \frac{e^y - e^{-y}}{e^y + e^{-y}} = x$$

$$1 - e^{-2y} = x(1 + e^{-2y}) \implies e^{-2y}(1 + x) = 1 - x$$

$$e^{2y} = \frac{1+x}{1-x} \implies 2y = \ln\left(\frac{1+x}{1-x}\right) \implies y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

2. For $x \in]-1, -1[\cup [1, +1[$:

- Assume $arccoth x = y$.

$$\coth y = x \implies \frac{e^y + e^{-y}}{e^y - e^{-y}} = x$$

$$\frac{1 + e^{-2y}}{1 - e^{-2y}} = x \implies 1 + e^{-2y} = x(1 - e^{-2y})$$

$$e^{-2y}(1 + x) = x - 1 \implies e^{2y} = \frac{1 + x}{x - 1}$$

$$2y = \ln\left(\frac{1 + x}{x - 1}\right) \implies y = arccoth x = \frac{1}{2}\ln\left(\frac{1 + x}{x - 1}\right)$$

3. **For $x \in \mathbb{R}$:**

- Assume $arcsinh x = y$.

$$\sinh y = x$$

- Using the identities:

$$e^y = \sinh y + \cosh y \quad \text{and} \quad \cosh y = \sqrt{\sinh^2 y + 1}$$

- We have:

$$e^y = x + \sqrt{x^2 + 1} \implies y = arcsinh x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

4. For $x \geq 1$:

- Assume $arccoshx = y$.

$$\cosh y = x$$

- Using the identities:

$$e^y = \cosh y + \sinh y \quad \text{and} \quad \sinh y = \sqrt{\cosh^2 y - 1}$$

- We have:

$$e^y = x + \sqrt{x^2 - 1} \implies y = arccoshx = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Thanks