

# Linear programming (LP)

- Year 2025/2026
  - Dr. Guettiche Mourad
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# I Duality

- In this chapter, we will see how, starting from a given linear program (which will be called the primal program), we can construct another linear program called the dual program.
- There are close links between these two programs: if one of them has an optimal solution, the other also has one, and the optimal values of the two programs coincide.

## II. Canonical form

- The canonical form is a transformation that harmonizes the writing of a PL.
- It can take two distinct forms:  
searching for a maximum or searching for a minimum.

## II. Canonical form

- Maximization problem:

The objective function is to be maximized.

$$\text{Max } Z = \sum_{j=1}^n c_j x_j$$

All constraints are of the less than or equal type ( $\leq$ ).

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, i=1, \dots, m$$

All decision variables must be non-negative.  $x_j \geq 0, j=1, \dots, n$

## II. Canonical form

- Minimization problem:

The objective function is to be minimized

$$\text{Min } Z = \sum_{j=1}^n c_j x_j$$

All constraints are of type greater than or equal to ( $\geq$ ).

$$\sum_{j=1}^n a_{ij} x_j \geq b_i, i=1, \dots, m$$

All decision variables must be non-negative.  $x_j \geq 0, j=1, \dots, n$

# III. The construction of the dual program

$$\text{Max } Z = CX$$

$$\text{s.t. } \begin{cases} AX \geq b \\ X \geq 0 \end{cases}$$

$$\text{With } A = (a_{ij}) \in M_{mn}, b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}, C = [c_1, c_2, \dots, c_n], X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{pmatrix}$$

# III. The construction of the dual program

$$\text{Min } Y=Yb$$

$$s.c. \begin{cases} YA \geq C \\ Y \geq 0 \end{cases}$$

$$Y = [y_1, y_2, \dots, y_m]$$

# III.1 Primal vs Dual

Primal	dual
Maximisation	Minimisation
N variables	N constraints
M constraints	M variables
variable $n^o j \geq 0$	contraintes $n^o j \geq 0$
variable $n^o j \leq 0$	contraintes $n^o j \leq 0$

# III.1 Primal vs Dual

Primal	dual
contrainte $n^o i \leq$	Variable $n^o i \geq 0$
contrainte $n^o i \geq$	Variable $n^o i \leq 0$
contrainte $n^o i =$	Variable $n^o i \in \mathcal{R}$
Objectif function coefficients $C_j$	second membre $b_i$
second membre $b_i$	Objectif function coefficients $C_j$

# IV Duality theorems

- An essential property of the dual problem is that any admissible solution to the dual provides an upper bound on the optimal value of the primal.

## **Theorem 1. (Weak duality)**

- Let  $Y$  be a feasible solution to the dual and  $x$  be a feasible solution to the primal. Then we have:

$$Z = CX \leq Yb = W$$

# IV Duality theorems

- **Corollary 1.** Let  $x^*$  be a feasible solution of the primal and  $y^*$  a feasible solution of the dual.

If  $Cx^* = \lambda^*b$ , then  $x^*$  is an optimal solution of (P) and  $\lambda^*$  an optimal solution of (D).

**Theorem 2.** (Strong duality) If the primal problem (P) has an optimal solution  $x^*$ , then the dual problem (D) has an optimal solution  $y^*$  and the optimal values coincide:

$cx^* = y^*b$ , in other words  $z^* = w^*$ .

# IV.1 Complementary Slackness Theorem

- Definition. let  $x^*$  be an optimal feasible solution of (P)

$$S.c. \begin{cases} AX \leq b \\ CX = Z(\max) \\ X \geq 0 \end{cases}$$

- The  $i^{\text{th}}$  constraint of linear programming will be said to be tight if  $A_i x^* = b_i$ , it will be said to be relaxed if  $A_i x^* < b_i$ .

# IV.1 Complementary Slackness Theorem

- A NSC for a pair of feasible solutions (P) and (D) to be an optimal solution is that:
  - 1- If a constraint of the linear program is relaxed, the corresponding variable of the dual is zero.
  - 2- If a constraint of the linear program is tight, the corresponding variable of the dual is zero.

# Application

$$\text{Max } Z = x_1 - 2x_2 + 3x_3$$

$$\begin{cases} x_1 + x_2 - 2x_3 \leq 1 \\ 2x_1 - x_2 - 3x_3 \leq 4 \\ x_1 + x_2 + 5x_3 \leq 2, \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Check the optimality of the solution:

$$x_1^* = 9/7, x_2^* = 0, x_3^* = 1/7$$

# V. Primal-dual correspondence

Primal	dual
Having an Optimal Solution	Having an Optimal Solution
unfeasible	unfeasible or having unbounded solutions
having unbounded solutions unbounded	unfeasible

# VI. Economic interpretation

- Primal and dual are two different views of the same problem.
- The primary objective concerns the allocation of resources: making the best use of the raw materials available through activities in order to maximize yield.
- The dual, meanwhile, expresses the price formation aspect: it analyses the importance of the values associated with raw materials in determining yield.
- The values of the dual variables are called marginal costs (or 'shadow prices') i.e the dual variable is the marginal cost of each unit of production.

## V.II Post-optimization and sensitivity analysis

- The rate of increase of the objective function with respect to the  $i^{\text{th}}$  component of the right-hand side of the primal problem is equal to the value of the  $i^{\text{th}}$  dual variable.
- Note that this interpretation has a validity limit: it is valid as long as the same basic set remains optimal.
- Note that if a constraint is unsaturated, then thanks to complementarity, the corresponding dual variable is zero, which shows that the marginal value is zero.

# Application

Consider an example problem where  $x_1$  and  $x_2$  represent the quantities of two products,  $P_1$  and  $P_2$ . Their respective margins are 10 and 9. The constraints represent production constraints, let us assume that the first constraint is a constraint on the stock of raw materials and the second is a constraint on the number of working hours per day. More specifically, to produce one unit of product  $P_1$ , three units of raw material and four hours of labor are used, while for one unit of  $P_2$ , three units of raw material and three hours of labor are needed. The stock is 11 units and the number of working hours per day is 24.

# Application

So we have the following linear program:  $\text{Max } z = 10x_1 + 9x_2$

$$S.c \begin{cases} 3x_1 + 3x_2 \leq 21 \\ 4x_1 + 3x_2 \leq 24 \\ x_1, x_2 \geq 0 \end{cases}$$

- If we increase the second member of the first constraint from 21 to 22, what will be the optimal solution below this value? Compute The rate of increase of the objective function ( $\Delta Z$ )

# Application

BV	x1	x2	x3	x4	sm
x3	3	3	1	0	21
x4	4	3	0	1	24
Z	10	9	0	0	0
Bv	x1	x2	x3	x4	sm
x3	0	$3/4$	1	$-3/4$	3
x1	1	$3/4$	0	$1/4$	6
z	0	$3/2$	0	$-5/2$	-60
Bv	x1	x2	x3	x4	sm
x2	0	1	$1/2$	-1	4
x1	1	0	-6	1	3
Z	0	0	-2	-1	-66

Thank you !