Chapter 3

Numerical series

3.1 Series with Real Terms

Definition 3.1.1: Let (u_n) be a sequence with real numbers. We call the infinite sum

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n \geqslant 0} u_n$$

a *numerical series*, and u_n is called the *general term* of the series.

If we put

$$S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} u_k,$$

then (S_n) is called the *sequence of partial sums* of the series $\sum_{n \geqslant 0} u_n$.

Example 3.1.1

1. *The harmonic series:*

$$\sum_{n \geqslant 1} \frac{1}{n}.$$

2. *The geometric series:*

$$\sum_{n \geqslant 0} ar^n = a + ar + \cdots + ar^n$$

Definition 3.1.2: A series with real terms $\sum_{n \geq 0} u_n$ is said to be **convergent** if the sequence of partial sums $(S_n)_{n \geq 0}$ converges to a limit S, called the **sum of the series**.

$$S = \lim_{n \to +\infty} S_n = \sum_{n \geq 0} u_n.$$

Remark 3.1.1 *A numerical series that is not convergent is said to be* ***divergent****.*
($\lim_{n\to+\infty} S_n = \infty$ *or does not exist)*

Example 3.1.2 ❶ *Let* $\sum_{n\geq 1} u_n$ *be the series with general term*

$$u_n = \frac{1}{n(n+1)}, \quad n \geq 1.$$

The term u_n *can be rewritten as*

$$u_n = \frac{1}{n} - \frac{1}{n+1}, \quad n \geq 1.$$

The partial sum is

$$S_n = \sum_{k=1}^{n} u_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Thus,

$$S_n = 1 - \frac{1}{n+1}.$$

Taking the limit,

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore, the series $\sum_{n\geq 1} u_n$ *is convergent and its sum is* $S = 1$*.*

❷ *Let* $\sum_{n\geq 1} u_n$ *be the series with general term*

$$u_n = \ln\left(1 + \frac{1}{n}\right).$$

The term u_n *can be rewritten in the form*

$$u_n = \ln\left(1 + \frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n).$$

Let

$$S_n = \sum_{k=1}^{n} u_k.$$

Then

$$\begin{aligned} S_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) \\ &= \ln(n+1) - \ln(1). \end{aligned}$$

Since $\ln(1) = 0$, *we obtain*

$$S_n = \ln(n+1).$$

Thus,

$$\lim_{n\to+\infty} S_n = +\infty.$$

Therefore, the series

$$\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right)$$

diverges*.*

Proposition 3.1.1 *If the series* $\sum u_n$ *is convergent, then*

$$\lim_{n\to+\infty} u_n = 0.$$

The receprocal is false:

$$\left(\lim_{n\to+\infty} u_n = 0\right) \nRightarrow \left((\sum u_n)_n \textit{ is convergent}\right).$$

Remark 3.1.2 *If*

$$\lim_{n\to+\infty} u_n \neq 0,$$

then the series $(\sum u_n)_n$ *is* ***divergent****.*

Example 3.1.3

1) $\sum_{n=0}^{+\infty} \frac{n}{n+1}$, *we have*

$$u_n = \frac{n}{n+1} \xrightarrow[n\to+\infty]{} 1 \neq 0.$$

Then $\sum_{n=0}^{+\infty} \frac{n}{n+1}$ *diverges.*

2) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$ *we have*

$$u_n = \ln\left(1+\frac{1}{n}\right) \xrightarrow[n\to+\infty]{} 0.$$

and, $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$ *is divergent (see exemple 3.1.2).*

Particular series

1/ **Riemann serie:** $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$, converges iff $\alpha > 1$.

2/ **harmonic serie:** $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.

3/ **geometric serie** $\sum_{n=0}^{+\infty} r^n$, converges iff $|r| < 1$.

3.1.1 Operations on Numerical Series

Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical series, then

- If $\sum_{n\geq 0} u_n$ converges to S_1 and $\sum_{n\geq 0} v_n$ converges to S_2, then

$$\sum_{n\geq 0} (u_n + v_n) \text{ converges to } S_1 + S_2.$$

- If $\sum_{n\geq 0} u_n$ converges to S and $a \in \mathbb{R}$, then

$$\sum_{n\geq 0} a u_n \text{ converges to } aS.$$

- If $\sum_{n\geq 0} u_n$ converges and $\sum_{n\geq 0} v_n$ diverges, then

$$\sum_{n\geq 0} (u_n + v_n) \text{ diverges.}$$

- If both series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} u_n$ diverge, nothing can be concluded about the nature of the series

$$\sum_{n\geq 0}(u_n+v_n).$$

Example 3.1.4 *Let the series*

$$\sum_{n\geq 0}\left(\frac{1}{5^n}+2^n\right) \quad \text{and} \quad \sum_{n\geq 0}\left(\frac{1}{5^n}-2^n\right).$$

Put

$$u_n=\frac{1}{5^n}+2^n, \qquad v_n=\frac{1}{5^n}-2^n.$$

Then

$$u_n+v_n=\frac{2}{5^n}.$$

Hence,

$$\sum_{n\geq 0}(u_n+v_n)=\sum_{n\geq 0}\frac{2}{5^n},$$

which is a geometric series with ratio $q=\frac{1}{5}<1$*, and therefore convergent.*

and we have also

$$u_n-v_n=2^{n+1}.$$

Then

$$\sum_{n\geq 0}(u_n-v_n)=\sum_{n\geq 0}2^{n+1},$$

which is divergent (geometric serie $q=2>1$*)*

3.2 Series with Positive Terms

Definition 3.2.1: A series $\sum u_n$ is called a *series with positive terms* if

$$u_n\geq 0 \quad \text{for all } n\in\mathbb{N}.$$

Proposition 3.2.1 *Let* $\sum u_n$ *be a series with positive terms. Then*

$$\sum u_n \text{ converges to } s \iff S_n \text{ is bounded } (S_n\leqslant s \text{ for all } n\in\mathbb{N}),$$

where $S_n=\sum_{k=0}^{n}u_k$.

3.2.1 Comparison Test

Theorem 3.2.1 (Comparison Rule) *Let $(\sum u_n)_n$ and $(\sum v_n)_n$ be two series with positive terms. Assume that there exists $n_0 \in \mathbb{N}$ such that*

$$u_n \leq v_n \quad \text{for all } n \geq n_0.$$

- *If $\sum_{n\geqslant 0} v_n$ converges, then $\sum_{n\geqslant 0} u_n$ converges.*
- *If $\sum_{n\geqslant 0} u_n$ diverges, then $\sum_{n\geqslant 0} v_n$ diverges.*

Example 3.2.1 *Consider the series*

$$\sum_{n=0}^{+\infty} \sin\left(\frac{1}{2^n}\right).$$

Since

$$0 < \sin\left(\frac{1}{2^n}\right) \leq \frac{1}{2^n},$$

and since $\sum \frac{1}{2^n}$ is a convergent geometric series, then the given series converges by comparison.

Corollary 3.2.1 *Let $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ be two series with positive terms, if there exists $a, b > 0$ satisfying*

$$au_n \leq v_n \leq bu_n$$

then $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$ are the same nature.

3.2.2 Tests of convergence

Theorem 3.2.2 (Test of Cauchy) *Let $\sum_{n\geqslant 0} u_n$ be a series with strictly positive terms and if*

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} (u_n)^{\frac{1}{n}} = \ell$$

exists.

- *If $\ell < 1$, then $\sum_{n\geqslant 0} u_n$ is convergent.*

- *If $\ell > 1$, then $\sum_{n\geqslant 0} u_n$ is divergent.*
- *If $\ell = 1$, then we can't say any thing.*

Example 3.2.2

1. $\sum_{n\geqslant 0} \left(\frac{n+5}{2n+1}\right)^n$

   We have
   $$\sqrt[n]{u_n} = \frac{n+5}{2n+1} \xrightarrow[n\to+\infty]{} \frac{1}{2} < 1.$$
   Then, the serie is converges.

2. $\sum_{n\geqslant 0} \left(\frac{2}{3}\right)^{n^2}$ *We have*
   $$\sqrt[n]{u_n} = \left(\frac{2}{3}\right)^n \xrightarrow[n\to+\infty]{} 0 < 1.$$
   Then, the serie $\sum_{n\geqslant 0} \left(\frac{2}{3}\right)^{n^2}$ is converges.

Theorem 3.2.3 (Test of D'Alembert) *Let $\sum_{n\geqslant 0} u_n$ be a series with positive terms. If*
$$\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \ell$$
exists, then:

- *If $\ell < 1$, the serie $\sum_{n\geqslant 0} u_n$ converges.*
- *If $\ell > 1$, the serie $\sum_{n\geqslant 0} u_n$ diverges.*

Example 3.2.3

1. $\sum_{n\geqslant 0} \frac{2^n}{n!}$.

   We compute
   $$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2}{n+1} \xrightarrow[n\to+\infty]{} 0 < 1.$$
   Therefore, the serie $\sum_{n\geqslant 0} \frac{2^n}{n!}$ is converges.

2. $\sum_{n\geqslant 0} \frac{(n+1)!}{n^2}$. *We compute*
   $$\left|\frac{u_{n+1}}{u_n}\right| = (\frac{n}{n+1})^2.(n+2) \xrightarrow[n\to+\infty]{} +\infty.$$

Therefore, the serie $\sum_{n\geqslant 0} \frac{(n+1)!}{n^2}$. *is diverges.*

Theorem 3.2.4 (Riemann's Test) *Let* $\sum u_n$ *be a series with positive terms and* $\alpha \in \mathbb{R}$ *such that*

$$\lim_{n\to+\infty} n^\alpha u_n = \ell.$$

- *If* $\ell = 0$ *and* $\alpha > 1$, *then* $\sum u_n$ *converges.*
- *If* $\ell = +\infty$ *and* $\alpha \leq 1$, *then* $\sum u_n$ *diverges.*
- *If* $\ell \neq 0 \neq +\infty$, *then* $\sum u_n$ *has the same nature as* $\sum \frac{1}{n^\alpha}$.

Example 3.2.4 *Let*

$$u_n = \frac{1}{n^2 \ln n}.$$

We have

$$\lim_{n\to+\infty} n^2 u_n = \frac{1}{\ln n} = 0.$$

Hence, the series

$$\sum_{n\geq 2} \frac{1}{n^2 \ln n}$$

convergent with $\alpha = 2 \geqslant 1$.

Chapter 4

Descriptive Statistics

Descriptive statistics is the set of scientific methods used to collect, describe and analyze observed data.

4.1 Statistical Vocabulary

Definition 4.1.1: Population: is the set of individuals or objects of the same nature on which the study relates.

Definition 4.1.2: Individuals or statistical units are the elements of the population.

Definition 4.1.3: Sample: is a subset of the population.

Definition 4.1.4: Statistical variable or character X is the subject under statistical study.

Definition 4.1.5: Statistical modality or category: the different possible situations (levels) of a statistical variable.

There are two types of statistical variables.

Definition 4.1.6: [Quantitative variables] Quantitative variables are variables that can be measured. They are characterized by numerical values.

A quantitative statistical variable can be:

- Continuous: when it can take numbers from an interval of real numbers.
- Discrete: if it takes isolated values.
- Temporal: variables that use units of measurement of time.

Example 4.1.1

Variable	*Possible modalities*	*Type of variable*
Height	*1.70m, 1.60m, 1.65m*	*Continuous quantitative*
Number of students	*30, 50, 60*	*Discrete quantitative*

Definition 4.1.7: [Qualitative variables] Qualitative variables are variables that are not measurable. Their modalities are words.

Qualitative statistical variables can be:

- Nominal
- Ordinal

Example 4.1.2

Variable	*Modalities*	*Type*
Eye color	*black, blue, green*	*Nominal qualitative*
Satisfaction	*satisfied, dissatisfied*	*Ordinal qualitative*

Definition 4.1.8: Statistical series: the simplest form of presenting statistical data relating to a single variable consists of a simple enumeration of the values taken by the variable.

Definition 4.1.9: Absolute frequency n_i is the number of statistical elements relating to a given modality.

Definition 4.1.10: Relative frequency $f_i = \frac{n_i}{n}$.

Example 4.1.3 *The marks of 9 students are:*

x_i	n_i	f_i	f_i^c
5	2	2/9	2/9
6	1	1/9	1/3
8	3	1/3	2/3
12	2	2/9	8/9
16	1	1/9	1

4.2 Position Parameters

Definition 4.2.1: Position parameters are values located in the center of the statistical distribution.

Definition 4.2.2: [Mean] For a discrete variable,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} n_i x_i$$

Example 4.2.1

$$\bar{x} = \frac{1}{8}(0 \times 2 + 1 \times 3 + 2 \times 1 + 3 \times 1 + 4 \times 1) = 1.5$$

Definition 4.2.3: [Mode] The mode is the most commonly occurring value.

Definition 4.2.4: [Median] The median is the value at the center of the ordered series.

4.3 Dispersion Parameters

Definition 4.3.1: [Range] The range is defined by

$$e = x_{\text{max}} - x_{\text{min}}.$$

Definition 4.3.2: [Variance]

$$V(X) = \frac{1}{n}\sum_{i=1}^{k} n_i(x_i - \bar{x})^2.$$

Definition 4.3.3: [Standard deviation]

$$\sigma_X = \sqrt{V(X)}.$$

4.4 Shape Parameters

Definition 4.4.1: [Skewness] Pearson's skewness coefficient is defined by

$$A_P = \frac{\bar{x} - Mo}{\sigma_X}.$$

Definition 4.4.2: [Kurtosis] Fisher's kurtosis coefficient is defined by

$$A_{PF} = \frac{m_4}{\sigma_X^4} - 3.$$