Chapter 2

real functions of several real variables

2.1 Real-Valued Functions of Several Variables

Definition 2.1.1: Real-Valued Functions of Several Variables

A real function of several variables (multivariable real-valued function) is a function

$$f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R},$$

$$(x_1, x_2, \ldots, x_n) \longmapsto f(x_1, x_2, \ldots, x_n).$$

For each $(x_1, x_2, \ldots, x_n) \in D$, the value of f is a real number $f(x_1, x_2, \ldots, x_n)$.

Example 2.1.1

- *The volume of a cylinder:* $V = \pi r^2 h$, *is a function of two variables:*

$$V = f(r, h).$$

- *The function*

$$f(x, y, z) = x^2 + y^2 + z^2$$

is a function of three variables.

2.2 Functions of Two Variables

Definition 2.2.1: A real-valued function of two variables is a function

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R},$$

$$(x, y) \longmapsto f(x, y).$$

The domain of definition of f, denoted by D_f, is the set of all points of $\mathbb{R}^2$ for which $f(x, y)$ is defined:

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \text{ is defined}\}.$$

Example 2.2.1 *1. Let*

$$f : \mathbb{R}^2 \to \mathbb{R},$$

$$(x, y) \longmapsto f(x, y) = \frac{2}{x^2 + y^2}.$$

Then

$$D_f = \mathbb{R}^2 \setminus \{(0, 0)\}.$$

2. Let

$$f : \mathbb{R}^2 \to \mathbb{R},$$

$$(x, y) \longmapsto f(x, y) = \sqrt{1 - x^2 - y^2}.$$

Then

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

D_f *is the disk with center* $(0, 0)$ *and radius* 1

2.2.1 Graphical Representation

Definition 2.2.2: The graph of a function f of two variables is defined by

$$\text{Graph}(f) = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D_f,\ z = f(x, y)\}.$$

This graph represents a surface in three-dimensional space.

Example 2.2.2

Let

$$f(x,y) = \sqrt{1 - x^2 - y^2}.$$

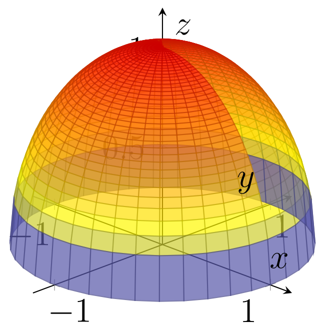
We write

$$z = f(x,y) = \sqrt{1 - x^2 - y^2}.$$

Then

$$z^2 + x^2 + y^2 = 1.$$

Hence, the graph of f is a hemisphere of radius 1.



2.2.2 Partial Derivatives

Definition 2.2.3:

Let $f(x,y)$ be a function of two variables.

- The partial derivative of f with respect to x at the point (x_0, y_0) written as $\frac{\partial f}{\partial x}(x_0, y_0)$ or $f'_x(x_0, y_0)$ is defined by

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

- The partial derivative of f with respect to y at the point (x_0, y_0) written as $\frac{\partial f}{\partial y}(x_0, y_0)$ or $f'_y(x_0, y_0)$ is defined by

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Example 2.2.3 *Using the definition, compute* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *for*

$$f(x,y) = x^2 y^3.$$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{(x+h)^2y^3 - x^2y^3}{h} = \lim_{h\to 0} \frac{2xhy^3 + h^2y^3}{h} = 2xy^3.$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{x^2(y+h)^3 - x^2y^3}{h} = \lim_{h\to 0} \frac{x^2 3y^2 h + 3yx^2h^2 + x^2h^3}{h} = 3x^2y^2.$$

so

$$\boxed{\frac{\partial f}{\partial x}(x,y) = 2xy^3, \qquad \frac{\partial f}{\partial y}(x,y) = 3x^2y^2}$$

2.3 Gradient

Definition 2.3.1:

Let $f(x,y)$ be a function such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (x_0, y_0).

The gradient of f at (x_0, y_0) is defined as

$$\operatorname{grad} f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right).$$

It is also denoted by

$$\nabla f(x_0,y_0).$$

Example 2.3.1 *Find the gradient $\nabla f(x,y)$ for each of the following functions.*

1. *Let*

$$f(x,y) = x^2 - xy + 3y^2$$

First order partial derivatives are:

$$\frac{\partial f}{\partial x}(x,y) = 2x - y$$

$$\frac{\partial f}{\partial y}(x,y) = -x + 6y$$

Thus, the gradient is:

$$\nabla f(x,y) = (2x - y, \; -x + 6y)$$

2. Let

$$f(x, y) = \sin(3x)\cos(3y)$$

Then:

$$\frac{\partial f}{\partial x}(x, y) = 3\cos(3x)\cos(3y)$$
$$\frac{\partial f}{\partial y}(x, y) = -3\sin(3x)\sin(3y)$$

Hence:

$$\nabla f(x, y) = (3\cos(3x)\cos(3y),\ -3\sin(3x)\sin(3y))$$

2.3.1 Second Order Partial Derivatives

Definition 2.3.2: Under conditions of existence, the **partial derivatives of order two** of a function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

are defined as the partial derivatives of the first-order derivatives.

If f_x and f_y exist, then:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right],$$
$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$
$$f_{xx} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right],$$
$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

Thus, there are four second-order partial derivatives.

Example 2.3.2 *Calculate the second partial derivatives for the function*

$$f(x, y) = xe^{-3y} + \sin(2x - 5y)$$

First derivatives:

$$f_x = \frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y)$$

$$f_y = \frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y)$$

Second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -4\sin(2x - 5y)$$
$$\frac{\partial^2 f}{\partial x \partial y} = -3e^{-3y} + 10\sin(2x - 5y)$$
$$\frac{\partial^2 f}{\partial y \partial x} = -3e^{-3y} + 10\sin(2x - 5y)$$
$$\frac{\partial^2 f}{\partial y^2} = 9xe^{-3y} - 25\sin(2x - 5y)$$

2.3.2 Differentials

Definition 2.3.3: Let f be a function of two variables with continuous first–order partial derivatives f_x and f_y. Then the differential of f is

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y.$$

Example 2.3.3 $f(x, y) = x^2y + 3xy^2$, *then*

$$\frac{\partial f}{\partial x} = 2xy + 3y^2, \qquad \frac{\partial f}{\partial y} = x^2 + 6xy,$$

so

$$\mathrm{d}f = (2xy + 3y^2)\,\mathrm{d}x + (x^2 + 6xy)\,\mathrm{d}y.$$

2.4 Double integrals

2.4.1 Double integrals over a rectangle

Let f be a continuous function on a rectangular domain $D = [a, b] \times [c, d]$. Then

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_a^b f(x, y)\,\mathrm{d}x \right) \mathrm{d}y$$
$$\text{by Fubini's theorem} = \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Example 2.4.1 *Evaluate*

$$I = \int_0^1 \int_0^1 (2x + y)\,\mathrm{d}x\,\mathrm{d}y.$$

First integrate with respect to x:

$$\int_0^1 (2x + y)\,\mathrm{d}x = \left[x^2 + xy\right]_0^1 = 1 + y.$$

Then integrate with respect to y:

$$I = \int_0^1 (1 + y)\,\mathrm{d}y = \left[y + \frac{y^2}{2}\right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Remark 2.4.1 *If $f(x, y) = g(x)h(y)$ where $g : [a, b] \to \mathbb{R}$ and $h : [c, d] \to \mathbb{R}$ are continuous functions, then:*

$$\int_a^b \int_c^d f(x, y)dxdy = \int_a^b g(x)dx \int_c^d h(y)dy \tag{2.1}$$

2.4.2 Double Integrals over Non-Rectangular Domains

Let f be a continuous function over a domain $D \subseteq \mathbb{R}^2$. D can be represented by one of the following forms:

1. If $D = \{(x, y) \in \mathbb{R}^2, a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$, then:

$$\iint_D f(x, y)dxdy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy\right) dx$$

2. If $D = \{(x, y) \in \mathbb{R}^2, c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$, then:

$$\iint_D f(x, y)dxdy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx\right) dy$$

Example 2.4.2 *Evaluate $I = \iint_D xydxdy$*
where $D = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, x + y \leq 1\}$.

We have $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, 0 \leq y \leq 1 - x\}$. Then:

$$I = \iint_D xydxdy = \int_0^1 \left[\int_0^{1-x} xydy\right] dx$$

$$
\begin{aligned}
&= \int_0^1 \left[\frac{x}{2}y^2\right]_0^{1-x} dx = \int_0^1 \frac{x}{2}(1-x)^2 dx \\
&= \frac{1}{2}\int_0^1 (x^3 + x - 2x^2) dx = \frac{1}{2}\left[\frac{1}{4}x^4 + \frac{x^2}{2} - \frac{2}{3}x^3\right]_0^1 \\
&= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{2} - \frac{2}{3}\right] = \frac{1}{2}\left[\frac{3}{4} - \frac{2}{3}\right] \\
&= \frac{1}{2}\left[\frac{9-8}{12}\right] = \frac{1}{24}
\end{aligned}
$$

Triple Integrals

Triple integrals are an extension of double integrals to three dimensions.

Definition 2.4.1: A triple integral is an integral of a function of three variables over a three-dimensional region P. It is written as:

$$\iiint_P f(x, y, z)\, dx\, dy\, dz$$

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a parallelepiped $P = [a, b] \times [c, d] \times [e, f]$, then:

$$\iiint_P f(x, y, z)\, dx\, dy\, dz = \int_a^b \left(\int_c^d \int_e^f f(x, y, z)\, dz\, dy \right) dx$$

This integral is also equal to any of the five other possible orderings for the integration variables iterated Triple Integral: for a, b, c, d, e, f real numbers, so:

$$
\begin{aligned}
\int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz &= \int_e^f \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz \\
&= \int_c^d \int_e^f \int_a^b f(x, y, z)\, dx\, dz\, dy \\
&= \int_c^d \int_a^b \int_e^f f(x, y, z)\, dz\, dx\, dy \\
&= \int_a^b \int_e^f \int_c^d f(x, y, z)\, dy\, dz\, dx \\
&= \int_a^b \int_c^d \int_e^f f(x, y, z)\, dz\, dy\, dx
\end{aligned}
$$

Example 2.4.3 *Evaluate the triple integral:*

$$\int_0^1 \int_2^4 \int_{-1}^5 (x + yz^2)\, dx\, dy\, dz$$

We integrate with respect to x first, then y, and then z.

$$\begin{aligned}
I &= \int_0^1 \int_2^4 \left[\frac{x^2}{2} + xyz^2\right]_{-1}^5 dy\, dz \\
&= \int_0^1 \int_2^4 (12 + 6yz^2)\, dy\, dz \\
&= \int_0^1 \left[12y + 3y^2z^2\right]_2^4 dz \\
&= \int_0^1 (24 + 36z^2)\, dz \\
&= \left[24z + \frac{36}{3}z^3\right]_0^1 \\
&= 36
\end{aligned}$$

2.4.3 Triple Integrals Over a General Bounded Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region P is defined as:

$$P = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

In $\mathbb{R}^3$, this is:

$$\iiint_P f(x, y, z)\, dx\, dy\, dz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right) dxdy$$

Example 2.4.4 *Calculate*

$$I = \iiint_P dx\, dy\, dz$$

where $P = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x + y + 2z \leq 1\}$. We cut P by a horizontal plane $z = z_0$. We find a triangle D bounded by $x = 0, y = 0$ and $x + y = 1 - 2z_0$, such that $z_0 \in [0, \frac{1}{2}]$.

The limits are:

- $0 \leq z \leq \frac{1}{2}$

- $0 \leq x \leq 1 - 2z$
- $0 \leq y \leq 1 - 2z - x$

Then:

$$\begin{aligned}
I &= \int_0^{1/2} \int_0^{1-2z} \int_0^{1-2z-x} dy\, dx\, dz \\
&= \int_0^{1/2} \int_0^{1-2z} (1 - 2z - x)\, dx\, dz \\
&= \int_0^{1/2} \left(2z^2 - 2z + \frac{1}{2} \right) dz \\
&= \frac{1}{12}
\end{aligned}$$