Linear system resolution

iterative methods

Numerical Methods

16/12/2025

Introduction

Consider the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2,n-1}x_{n-1} + a_{2n}x_n = b_2, \\ \qquad\quad \vdots \quad + \quad \vdots \quad + \ddots \quad + \vdots \quad + \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1} + a_{nn}x_n = b_n. \end{cases}$$

The system matrix $A \in M_n(K)$, x= ????
Two types of methods
Direct methods VS Iterative methods

Direct vs Iterative Methods for Linear Systems

Criterion	**Direct Methods**	**Iterative Methods**
Solution type	Exact (up to round-off)	Approximate
Number of steps	Finite	Iterative (until convergence)
Initial guess	Not required	Required
Convergence	Always (if non-singular)	Depends on matrix properties
Matrix size	Small to medium	Large and sparse
Memory cost	High	Low
Examples	Gauss, LU, Cholesky	Jacobi, Gauss–Seidel

Also, iterative methods provide approximate solutions $x^{(k)}$ converging to the exact solution x for:

- Linear system solving
- Differential equation solving
- Optimization problems
- Roots of nonlinear equations
- Eigenvalue problems
- ...

Principles of Iterative Methods

1. Given the linear system:

$$Ax = b, \qquad \text{how can we express } x\,?$$

   The problem $Ax = b$ is reformulated as a fixed-point model:

$$x = f(x)$$

2. This type of problem is called a **fixed-point problem** or a **successive approximation problem**.

Principles of Iterative Methods

Construction of Iterative Sequences

3. Starting from an initial vector $x^{(0)}$, Iterative methods construct a sequence of vectors:

$$x^{(0)} \longrightarrow x^{(1)} \longrightarrow x^{(2)} \longrightarrow \cdots \longrightarrow x^{(k)}$$

3. Starting from an initial vector $x^{(0)}$, Iterative methods construct a sequence of vectors:

$$x^{(0)} \longrightarrow x^{(1)} \longrightarrow x^{(2)} \longrightarrow \cdots \longrightarrow x^{(k)}$$

Each sequence $\left(x^{(k)}\right)_{k\in\mathbb{N}}$, is a vector of unknowns $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})^T$, and is defined by:

$$\begin{cases} x^{(0)} \in \mathbb{R}^n \quad \text{(arbitrary initial guess)}, \\ x^{(k+1)} = f\big(x^{(k)}\big). \end{cases}$$

Principles of Iterative Methods

Construction of Iterative Sequences

3. Starting from an initial vector $x^{(0)}$, Iterative methods construct a sequence of vectors:

$$x^{(0)} \longrightarrow x^{(1)} \longrightarrow x^{(2)} \longrightarrow \cdots \longrightarrow x^{(k)}$$

Each sequence $\left(x^{(k)}\right)_{k\in\mathbb{N}}$, is a vector of unknowns $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})^T$, and is defined by:

$$\begin{cases} x^{(0)} \in \mathbb{R}^n & \text{(arbitrary initial guess)}, \\ x^{(k+1)} = f\big(x^{(k)}\big). \end{cases}$$

The sequence $\left(x^{(k)}\right)$ converges to the exact solution x of the system if:

$$\lim_{k\to\infty} \|x^{(k)} - x\| = 0$$

where $\|\cdot\|$ denotes a vector norm.

Fixed-Point Reformulation of the Linear System

4. By writing the matrix A in the form: $A = M - N$,

we obtain the following equivalences:

$$
\begin{aligned}
Ax = b &\iff (M - N)x = b \\
&\iff Mx - Nx = b \\
&\iff Mx = Nx + b \\
&\iff M^{-1}Mx = M^{-1}(Nx + b) \\
&\iff x = M^{-1}Nx + M^{-1}b \\
&\iff x = Bx + R,
\end{aligned}
$$

where $B = M^{-1}N$ and $R = M^{-1}b$.

- The solution vector X is now modeled by the iterative relation (the fixed-point model) $X^{(k+1)} = BX^{(k)} + R$

Fixed-Point Reformulation of the Linear System

4. By writing the matrix A in the form: $A = M - N$,

we obtain the following equivalences:

$$
\begin{aligned}
Ax = b &\iff (M-N)x = b \\
&\iff Mx - Nx = b \\
&\iff Mx = Nx + b \\
&\iff M^{-1}Mx = M^{-1}(Nx + b) \\
&\iff x = M^{-1}Nx + M^{-1}b \\
&\iff x = Bx + R,
\end{aligned}
$$

where $B = M^{-1}N$ and $R = M^{-1}b$.

- The solution vector X is now modeled by the iterative relation (the fixed-point model) $X^{(k+1)} = BX^{(k)} + R$
- $B = M^{-1}N$ is called the **iteration matrix**.

Main Decompositions of the System Matrix

The matrix A can be decomposed as:

$$A = D - E - F$$

where:

- D is a **diagonal matrix** (contains the diagonal elements of A),
- E is a **strictly lower triangular matrix** (without diagonal),
- F is a **strictly upper triangular matrix** (without diagonal).

These matrices are constructed directly from the elements of A.

Main Decompositions of the System Matrix

$$A = D - E - F \quad , D = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_{21} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & 0 & 0 \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ 0 & 0 & \cdots & -a_{2,n-1} & -a_{2n} \\ 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -a_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Main Decompositions of the System Matrix

$$A = D - E - F = \begin{pmatrix} & & -F \\ & D & \\ -E & & \end{pmatrix}$$

On the one hand, the matrix A can be written as:

$$A = M - N$$

On the other hand, we also have the decomposition:

$$A = D - E - F$$

Therefore, the matrices M and N can be obtained by different groupings of the matrices D, E, and F in the iterative model:

$$X^{(k+1)} = BX^{(k)} + M^{-1}b$$

- For the Jacobi method: $M = D \quad \text{and} \quad N = E + F$
  The iteration matrix is given by: $B = M^{-1}N = D^{-1}(E + F)$
- For the Gauss-Seidel method: $M = D - E \quad \text{and} \quad N = F + F$
  The iteration matrix is given by: $B = M^{-1}N = (D - E)^{-1}(F)$

Jacobi Decomposition

We consider the linear system: $Ax = b, \quad A \in \mathbb{R}^{4\times 4},\ x, b \in \mathbb{R}^4$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

The Jacobi method is based on the decomposition:

$$A = D - (E + F) \text{ where: } D = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -a_{21} & 0 & 0 & 0 \\ -a_{31} & -a_{32} & 0 & 0 \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} \\ 0 & 0 & -a_{23} & -a_{24} \\ 0 & 0 & 0 & -a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Jacobi Decomposition

The iteration matrix is given by: $B = D^{-1}(E + F)$.

Since D is diagonal, its inverse is: $D^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 & 0 \\ 0 & 0 & \frac{1}{a_{33}} & 0 \\ 0 & 0 & 0 & \frac{1}{a_{44}} \end{pmatrix}$

The Jacobi iteration matrix is:

$$B = \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 & 0 \\ 0 & 0 & \frac{1}{a_{33}} & 0 \\ 0 & 0 & 0 & \frac{1}{a_{44}} \end{pmatrix} \cdot \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & 0 & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & 0 & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{pmatrix}$$

Also, $R = D^{-1}b = \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 & 0 \\ 0 & 0 & \frac{1}{a_{33}} & 0 \\ 0 & 0 & 0 & \frac{1}{a_{44}} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$

$$B = \begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a_{14}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \frac{a_{24}}{a_{11}} \\ \frac{a_{31}}{a_{33}} & \frac{a_{32}}{a_{33}} & 0 & \frac{a_{34}}{a_{33}} \\ \frac{a_{41}}{a_{44}} & \frac{a_{42}}{a_{44}} & \frac{a_{43}}{a_{44}} & 0 \end{pmatrix}$$

and

$$R = \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \frac{b_3}{a_{33}} \\ \frac{b_4}{a_{44}} \end{pmatrix}$$

Element-by-Element Interpretation

The Jacobi update is: $\boxed{x^{(k+1)} = D^{-1}(E+F)x^{(k)} + D^{-1}b}$ where

$$x^{(k+1)} = \begin{pmatrix} x_1^{(k+1)} & x_2^{(k+1)} & x_3^{(k+1)} & x_4^{(k+1)} \end{pmatrix}$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - a_{24}x_4^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - a_{34}x_4^{(k)}\right)$$

$$x_4^{(k+1)} = \frac{1}{a_{44}}\left(b_4 - a_{41}x_1^{(k)} - a_{42}x_2^{(k)} - a_{43}x_3^{(k)}\right)$$

$$x^{(k+1)} = \begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a_{14}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \frac{a_{24}}{a_{11}} \\ \frac{a_{31}}{a_{33}} & \frac{a_{32}}{a_{33}} & 0 & \frac{a_{34}}{a_{33}} \\ \frac{a_{41}}{a_{44}} & \frac{a_{42}}{a_{44}} & \frac{a_{43}}{a_{44}} & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \\ x_4^{(k+1)} \end{pmatrix} + \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \frac{b_3}{a_{33}} \\ \frac{b_4}{a_{44}} \end{pmatrix}$$

Element-by-Element Interpretation

General term

$$x_i^{(k+1)} = \frac{b_i - \sum_{\substack{j=1,\\ i\neq j}}^{j=n} .a_{ij}x_i^{(k)}}{a_{ii}} \forall i = 1, \cdots, n$$

Convergence Conditions

The computed sequence is convergent if at least one of the following conditions is satisfied:

- **Strict diagonal dominance by rows**:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n$$

- **Spectral radius condition**:

$$\rho(M^{-1}N) < 1$$

where ρ denotes the spectral radius of the iteration matrix B:

$$\rho(B) = \max\{|\lambda| \mid \lambda \in \sigma(B)\}$$

Stopping Criteria

The iterative process is repeated until one of the following conditions is satisfied:

- A fixed number of iterations *NBR* is reached;
- The relative error satisfies:

$$\max_{1\leq i\leq n}\left|\frac{x_i^{(k+1)}-x_i^{(k)}}{x_i^{(k+1)}}\right|<\varepsilon,\quad \varepsilon>0$$

Example :

Consider the following system of linear equation:

$$\begin{cases} 4x_1 - x_2 + 2x_3 = 3 \\ 2x_1 - 6x_2 + 3x_3 = 4 \\ x_1 - 2x_2 + 4x_3 = 2 \end{cases}, \text{ its system matrix } \begin{bmatrix} 4 & -1 & 2 \\ 2 & -6 & 3 \\ 1 & -2 & 4 \end{bmatrix}$$

Solve using JACOBI method :

- Let $X^{(0)} = (0,0,0)^t$ the initial guess.
- Take the solution after 3 iterations

Solution

***Convergence of A is verified since it is a strictly diagonally dominant per row*.**

X	$X^{(0)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$
x_1	0	$x_1^{(1)} = \frac{1}{4}\left(3 + x_2^{(0)} - 2x_3^{(0)}\right)$	$x_1^{(2)} = \frac{1}{4}\left(3 + x_2^{(1)} - 2x_3^{(1)}\right)$	$x_1^{(3)} = \frac{1}{4}\left(3 + x_2^{(2)} - 2x_3^{(2)}\right)$
x_2	0	$x_2^{(1)} = \frac{-1}{6}\left(4 - 2x_1^{(0)} - 3x_3^{(0)}\right)$	$x_2^{(2)} = \frac{-1}{6}\left(4 - 2x_1^{(1)} - 3x_3^{(1)}\right)$	$x_2^{(3)} = \frac{-1}{6}\left(4 - 2x_1^{(2)} - 3x_3^{(2)}\right)$
x_3	0	$x_3^{(1)} = \frac{1}{4}\left(2 + x_1^{(0)} + 2x_2^{(0)}\right)$	$x_3^{(2)} = \frac{1}{4}\left(2 - x_1^{(1)} + 2x_2^{(1)}\right)$	$x_3^{(3)} = \frac{1}{4}\left(2 - x_1^{(2)} + 2x_2^{(2)}\right)$

$X^{(0)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$
0	$x_1^{(1)} = \frac{3}{4}$	$x_1^{(2)} = \frac{1}{3}$	$x_1^{(3)} = \frac{23}{32}$
0	$x_2^{(1)} = \frac{-2}{3}$	$x_2^{(2)} = \frac{-1}{6}$	$x_2^{(3)} = \frac{-163}{288}$
0	$x_3^{(1)} = \frac{1}{2}$	$x_3^{(2)} = \frac{-1}{48}$	$x_3^{(3)} = \frac{1}{3}$

Code MATLAB

```
function[x]=Jacobi(A,b)
n = length(b);
x0 = zeros(n, 1);
err = 1;
invM=inv(diag(diag(A)));
N= - (tril(A)+triu(A)-2*diag(diag(A)));
B=invM*N;
for cpt=1:7
    for i = 1 : n;
        x(i)=b(i);
        for j=1:n
            if i~=j
            x(i)=x(i)-A(i,j)*x0(j)
            end
        end
          x(i)=x(i)/A(i,i);
    end
 x'
x0 = x';
end
end
```