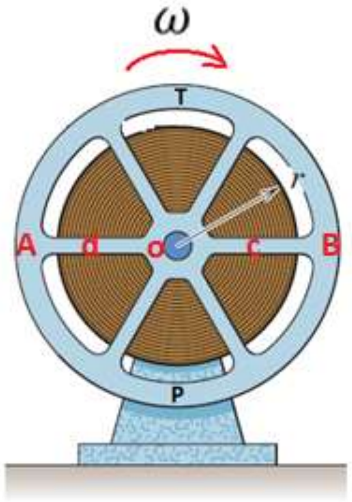
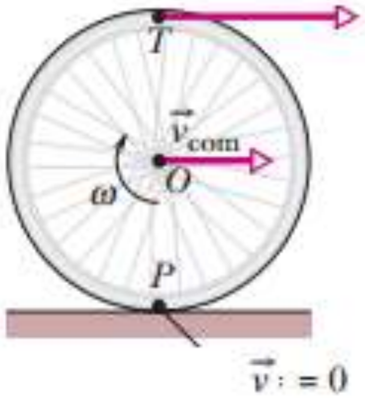
Kinematics of Rigid Bodies

Introduction

In Chapter 2 on particle kinematics, we developed the relationships governing the displacement, velocity, and acceleration of points as they moved along straight or curved paths. In rigid-body kinematics we use these same relationships but must also account for the rotational motion of the body. Thus rigid-body kinematics involves both linear and angular displacements, velocities, and accelerations, , and their variation with time.





A rigid body is a solid body in which deformation **is zero** or so small it can be neglected. The distance between any two given point on a rigid body remains constant in time regardless of external forces or moments exerted on it. A rigid body is usually considered as a continuous distribution of mass.

Characteristics of rigid body motion

All lines on a rigid body have the same angular velocity and the same angular acceleration. Rigid motion can be decomposed into the translation of an arbitrary point followed by a rotation about the point.

Motion

A body is said to be in motion if it **changes its position** with respect to a reference point A moving body can have three types of motions i.e., translational, rotational or general plane motion. The classification of motion is shown in figure 1.

Translation motion.

This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion. When the paths of motion for any two points on the body are parallel lines, the motion is called ***rectilinear translation***. If the paths of motion are along curved lines, the motion is called ***curvilinear translation***.

Rotation about a fixed axis.

When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along

circular paths.

General plane motion.

When a body is subjected to general plane motion, it undergoes a combination of translation and rotation. The translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.

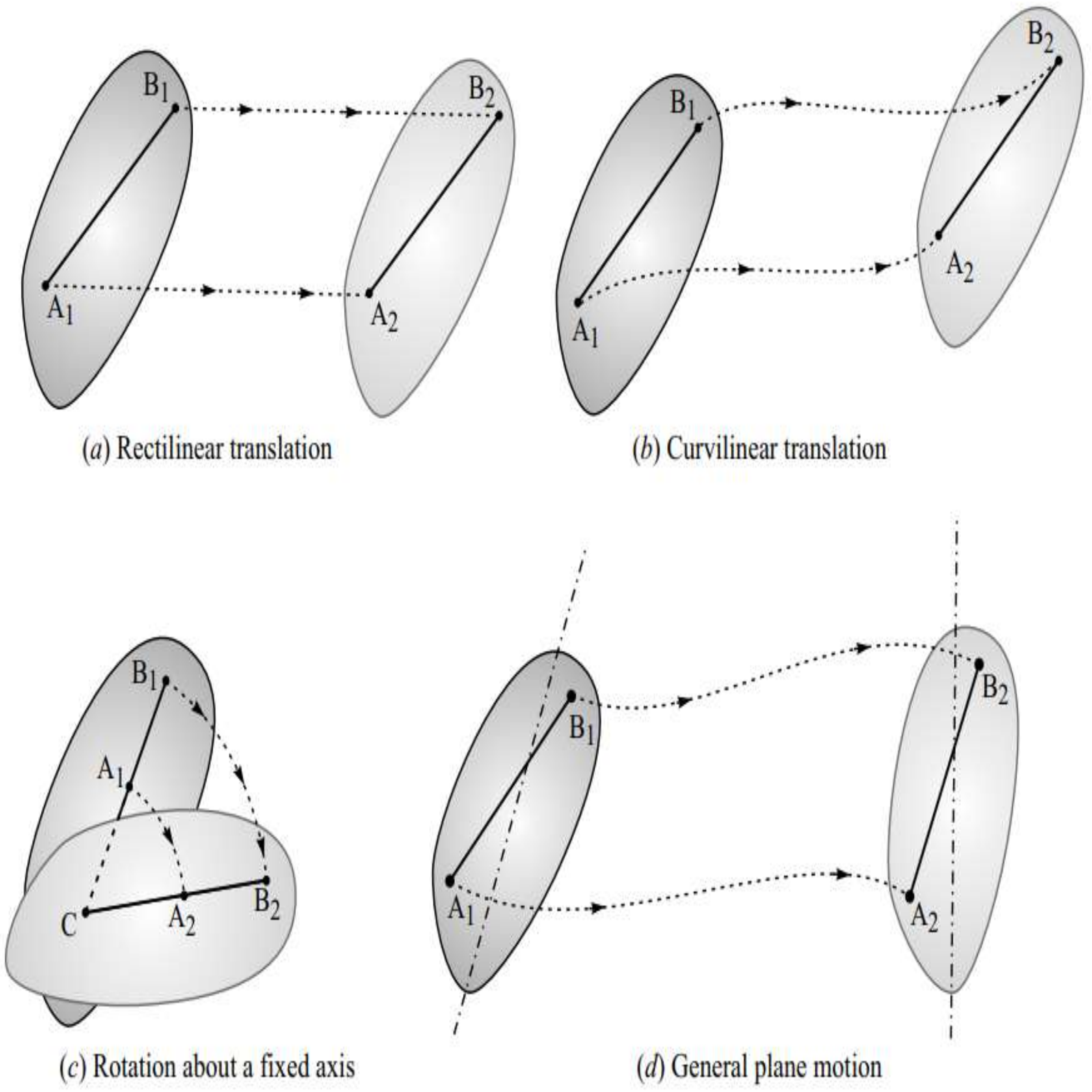


Fig.1.The classification of motion

To understand the general motion of a rigid body, we have taken a body of regular shape instead of an arbitrary shape as shown in Fig.2. We see that the body not only translates but also rotates in an arbitrary manner.

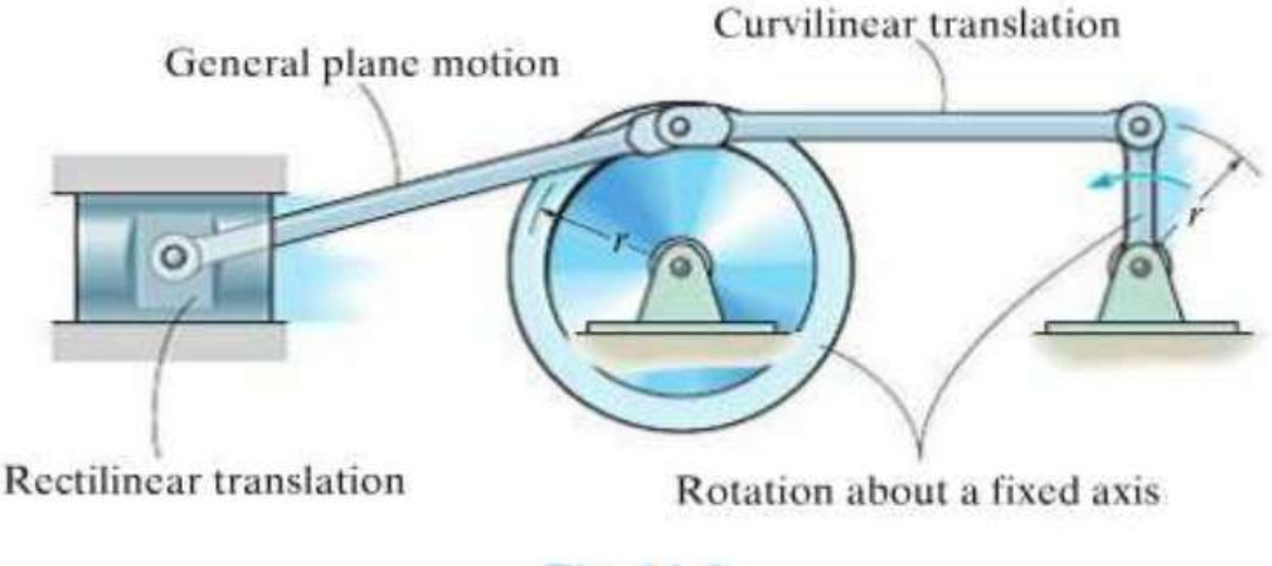


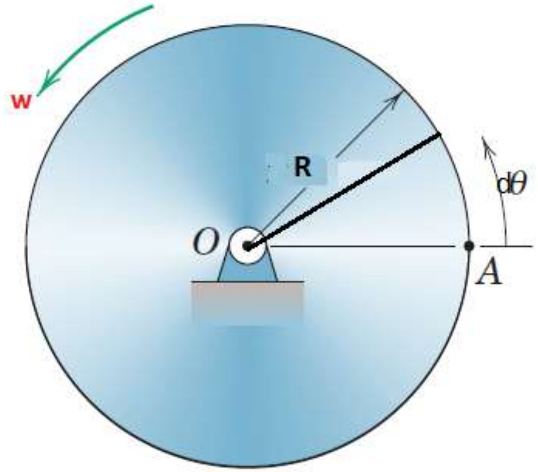
Fig.2.Diffrent motion of body.

Curvilinear

Rotational and Linear Kinematics

Rotational Motion	Quantity	Linear Motion
θ	Position	x
$\Delta\theta$	Displacement	Δx
ω	Velocity	v
α	Acceleration	a
t	Time	t

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

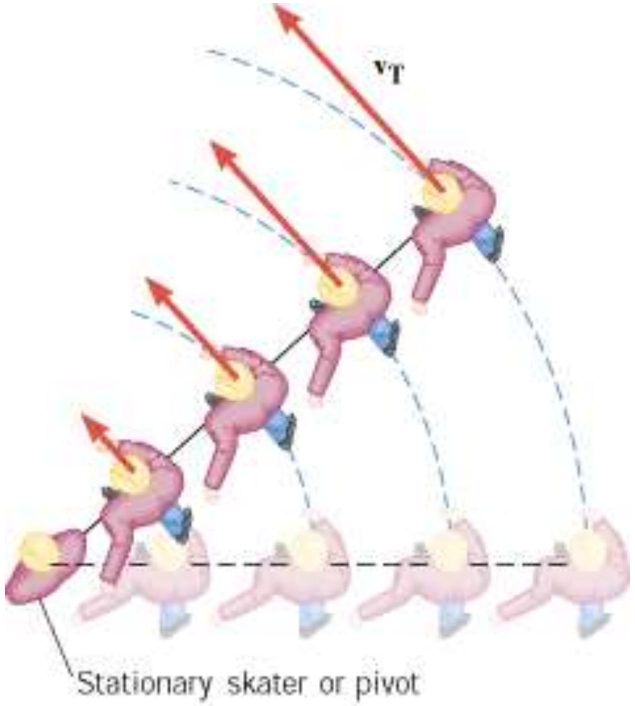


$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\omega \, d\omega = \alpha \, d\theta$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} \quad \text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Relating Rotational and Translational Speed



$$\left|\vec{v}\right| = v = \left|\omega\right| r \quad \text{(for radian measure only).}$$

- The rotational speed ω is the same at any points
- The translational speed is different for the points with different distance from the rotational axis.

Relative motion analysis

The position of a point on a moving rigid body in the 2D *xy*-plane

In planar kinematics, we fully determine the position of a rigid body by fixing the position vectors of 2 points in the rigid body that can be freely chosen, like points *A* and *B* of the rectangle in Fig.3.

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

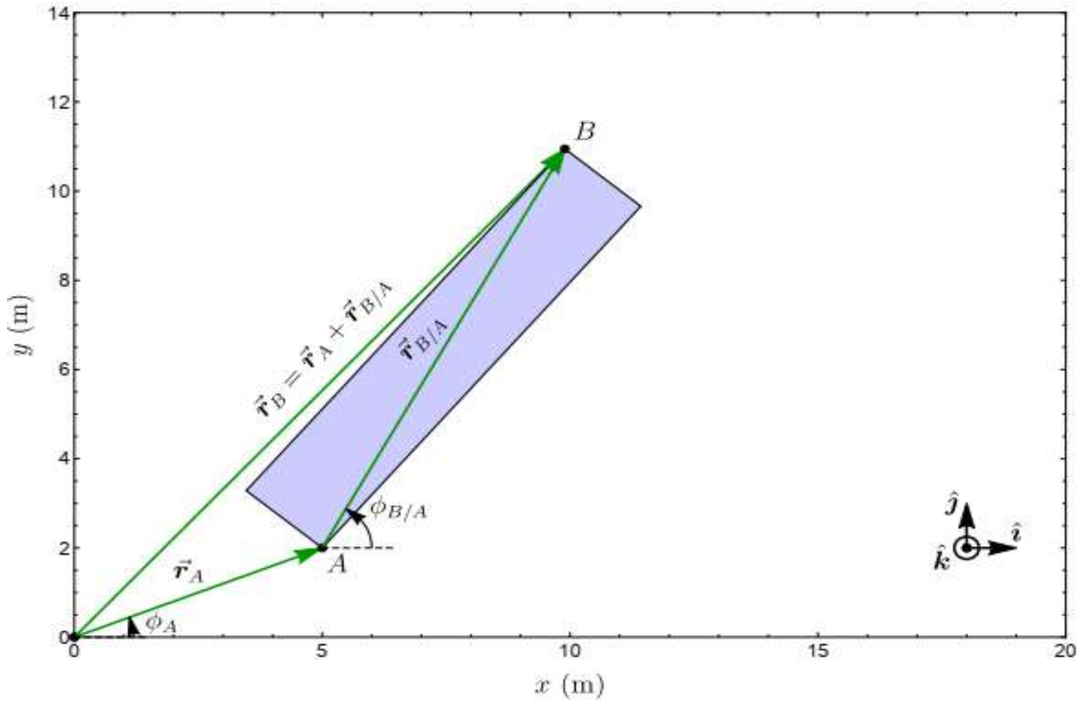


Figure .3 The orientation of a rigid body in the 2D *xy*-plane can uniquely be described by a position vector $\boldsymbol{r}_A$ and an angle $\phi_{B/A}$

Angular velocity vector

The angular velocity vector $\vec{\omega}$ a rigid body is a vector with magnitude $|\omega| = |\dot{\phi}_{B/A}|$

and a direction that is perpendicular to the plane in which the rigid body rotates. Its direction can be determined using the right hand rule.

The angular velocity vector (unit rad/s) of a rigid body that rotates in the *xy* plane is:

$$\vec{\omega}_{2D} = \omega\hat{\boldsymbol{k}} = \dot{\phi}_{B/A}\hat{\boldsymbol{k}}$$

We have 3 coordinates, namely x_A, y_A and $\phi_{B/A}$, where $\phi_{B/A}$ is the angle the relative position vector $\boldsymbol{r}_{B/A}$ makes with the x-axis. With x_A, y_A we can determine $\boldsymbol{r}_A$, and with $\phi_{B/A}$ and knowledge of the distance $|\boldsymbol{r}_{B/A}|$ we can determine $\boldsymbol{r}_{B/A}$:

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{A,2D} &= x_A\hat{\boldsymbol{\imath}} + y_A\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{B/A,2D} &= |\vec{\boldsymbol{r}}_{B/A}|\cos\phi_{B/A}\hat{\boldsymbol{\imath}} + |\vec{\boldsymbol{r}}_{B/A}|\sin\phi_{B/A}\hat{\boldsymbol{\jmath}}
\end{aligned}
$$

Now we determine $\boldsymbol{r}_B$ from the 3 coordinates by adding these two vectors as shown in

$$
\begin{aligned}
\vec{\boldsymbol{r}}_B &= \vec{\boldsymbol{r}}_A + \vec{\boldsymbol{r}}_{B/A} \\
\vec{\boldsymbol{r}}_{B,2D} &= (x_A + |\vec{\boldsymbol{r}}_{B/A}|\cos\phi_{B/A})\hat{\boldsymbol{\imath}} + (y_A + |\vec{\boldsymbol{r}}_{B/A}|\sin\phi_{B/A})\hat{\boldsymbol{\jmath}}
\end{aligned}
$$

Velocity of a point B in a rigid body

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B,2D} &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_B = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_A + \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\left(-\sin\phi_{B/A}\hat{\boldsymbol{\imath}} + \cos\phi_{B/A}\hat{\boldsymbol{\jmath}}\right) \\
&= \vec{\boldsymbol{v}}_A + \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A
\end{aligned}$$

The velocity of point B in can be split up in two parts: a vector $\boldsymbol{v}_{B,\mathrm{trans}}$ related to translation and a vector $\boldsymbol{v}_{B,\mathrm{rot}}$ related to rotation:

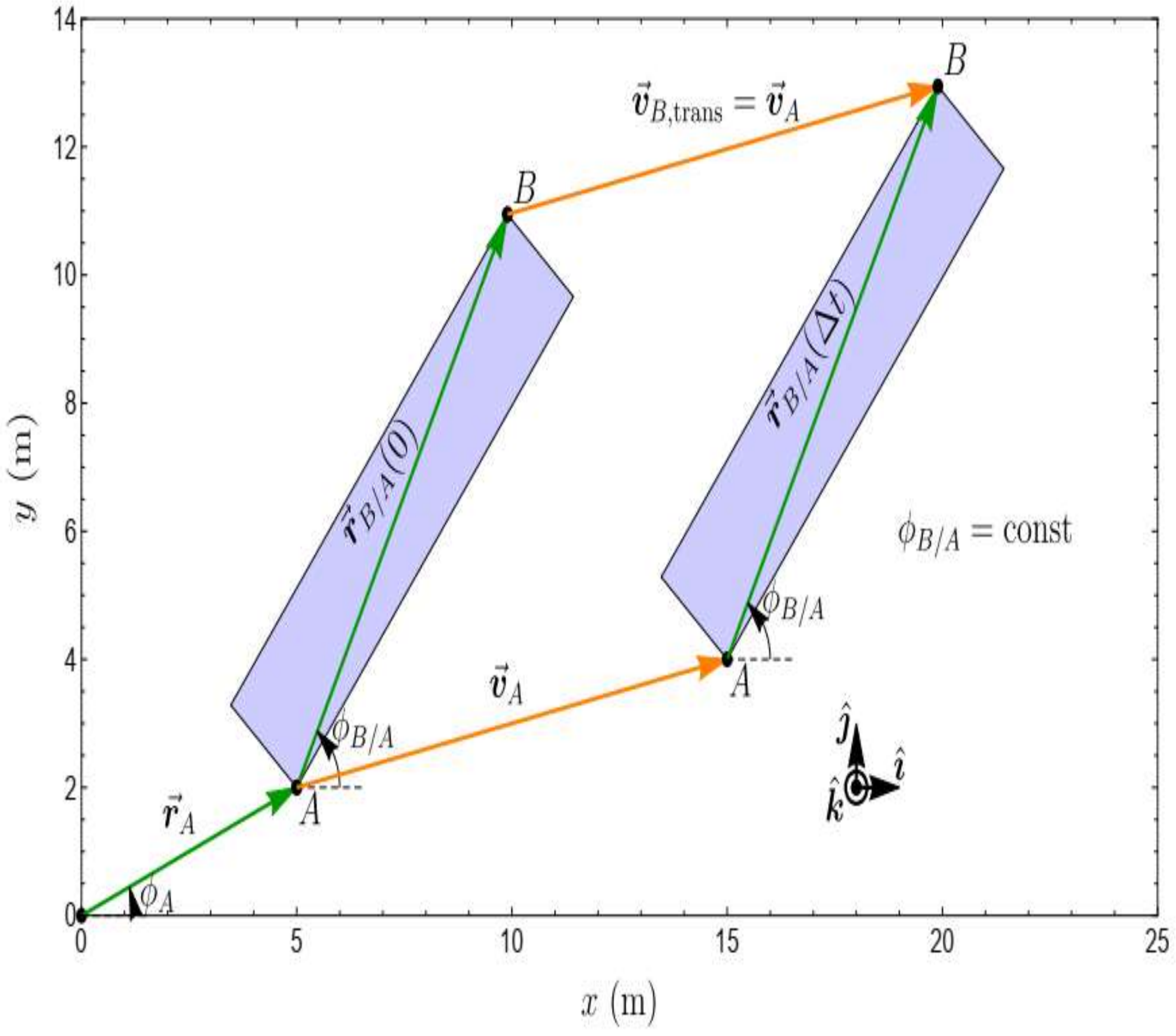


Figure .2: Pure translation of a rigid body: $\phi_{B/A}$ = constant .

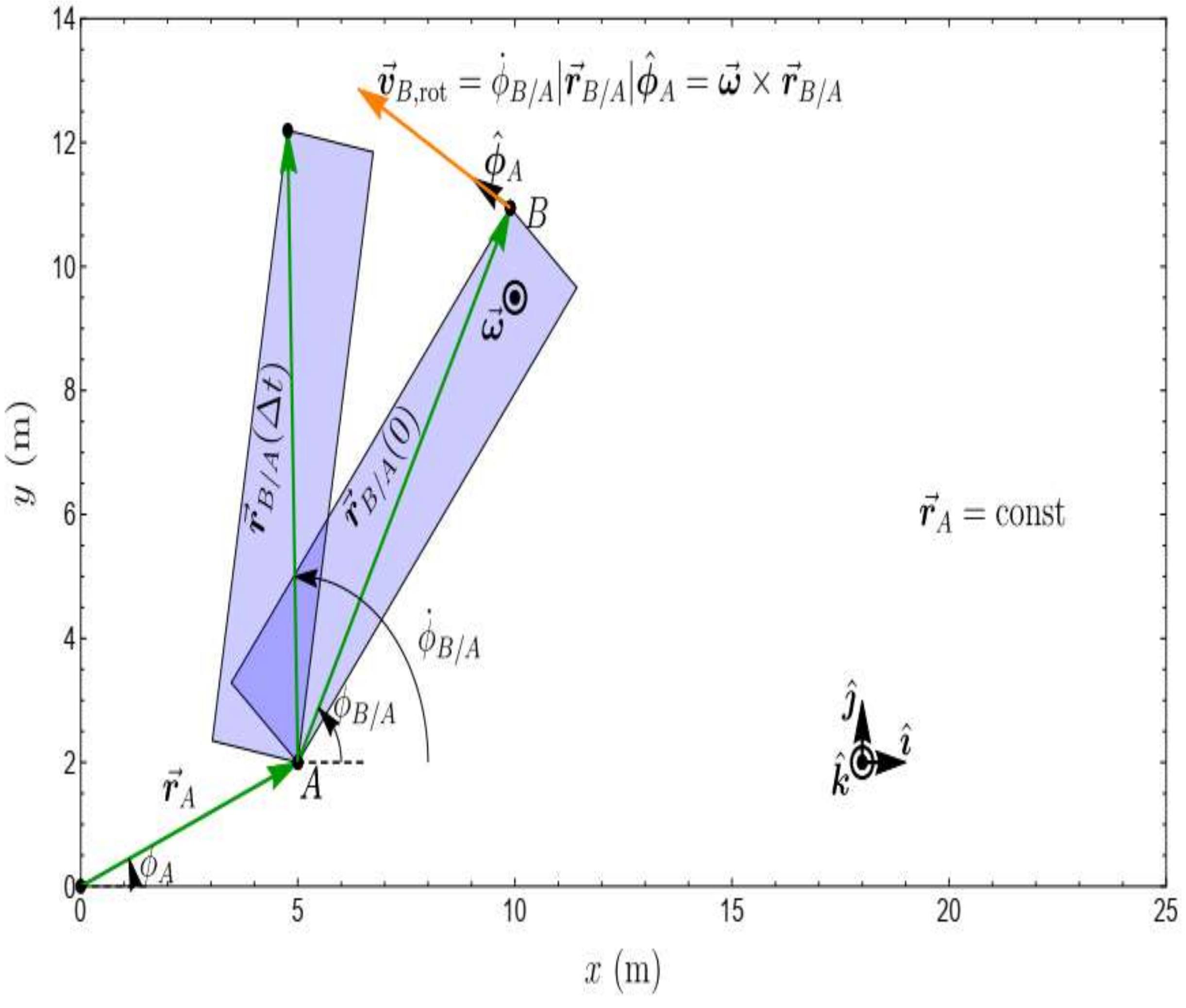
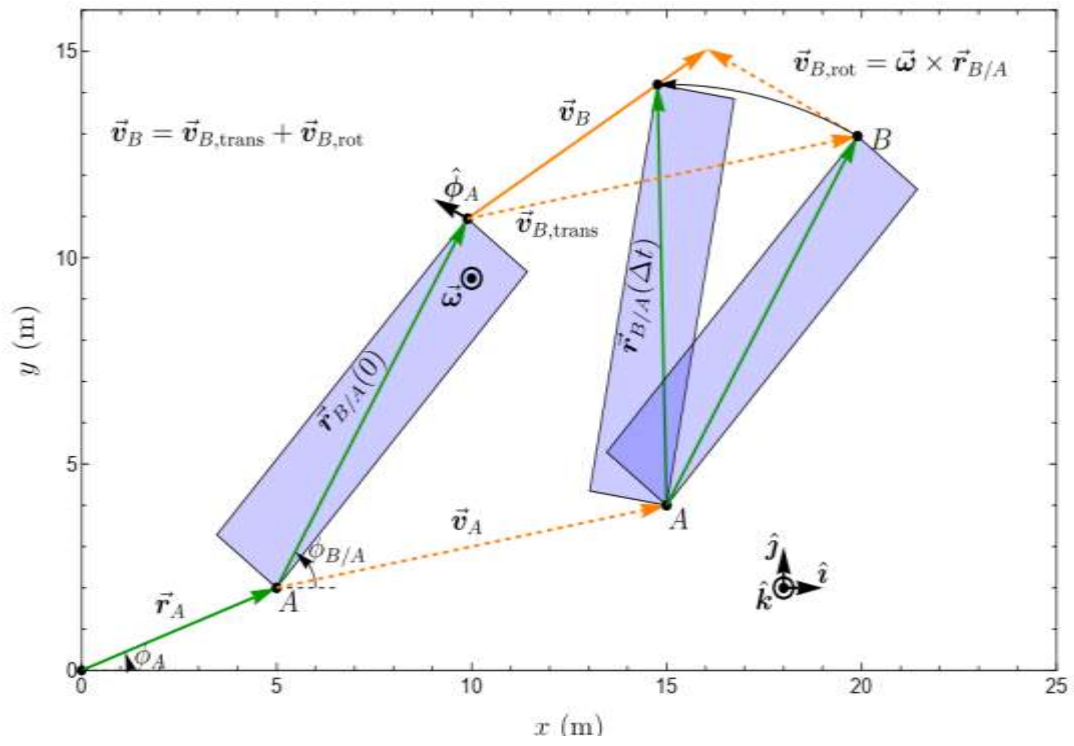


Figure .3: Pure rotation of a rigid body: $\boldsymbol{r}_A$ = constant.

General motion

In general, as shown in Fig.4, the motion of a point in a rigid body is a sum of translational and rotational motion.



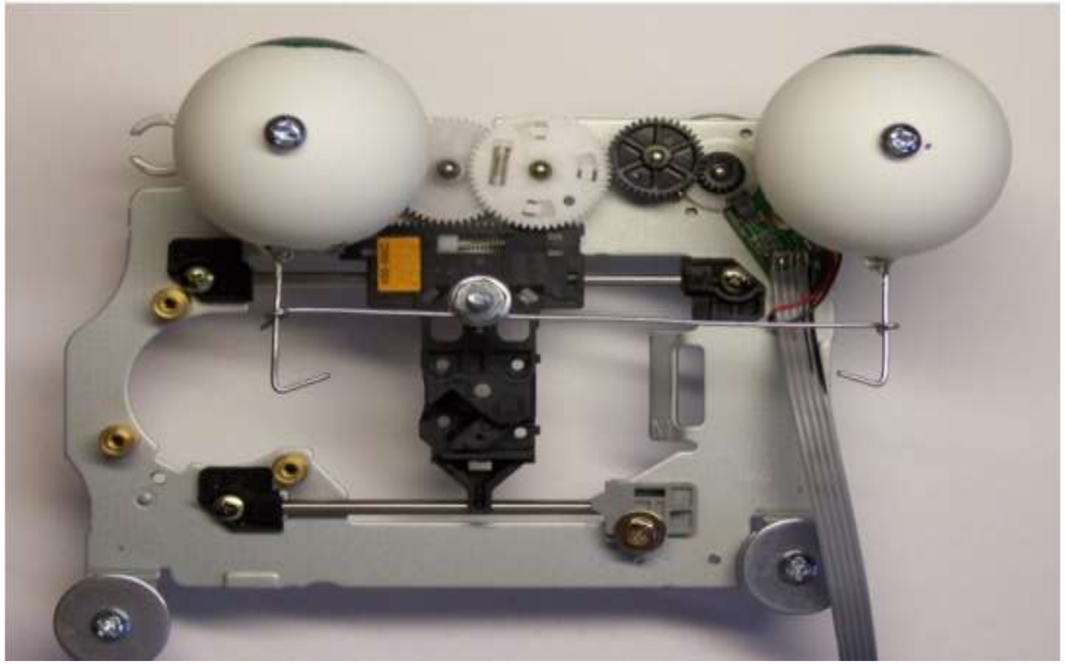
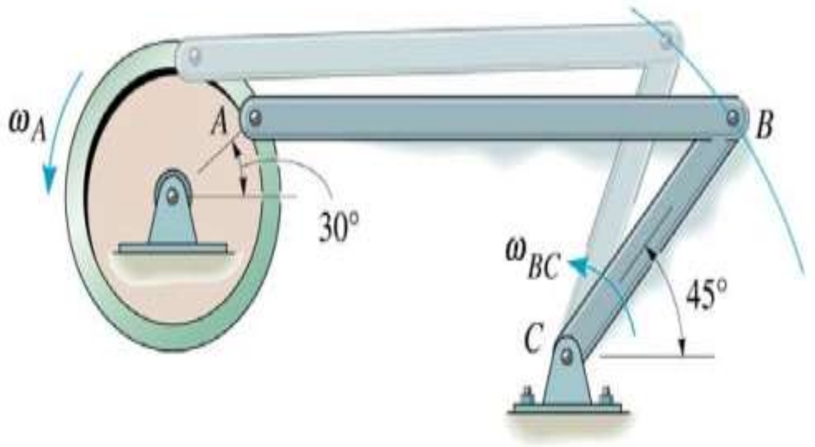


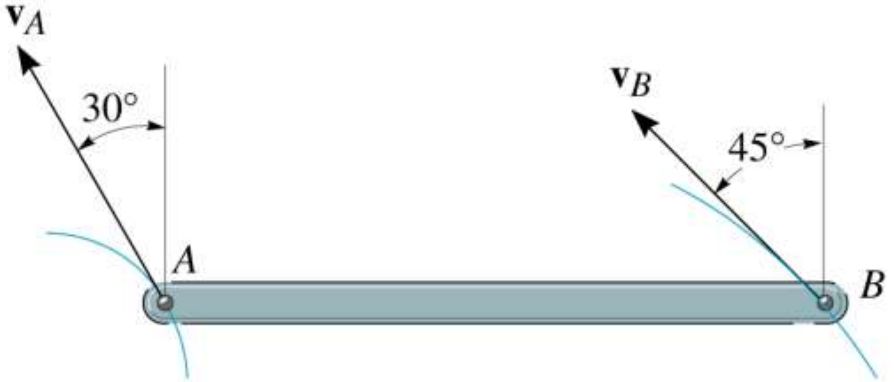
Figure .4: General motion of a rigid body: a combination of rotation and translation.

We obtain the most general equation and important equation for the velocity in a rigid body

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

Example When using the relative velocity equation. Points A and B should generally be points on the body with a know motion. Often these points are pin connections in linkages. Here both points A and B have circular motion since the disk and link BC move in circular paths. The direction vA and vB are known since they are always tangent to the circular path of motion.





Angular acceleration of a rigid body

We take the derivative of the velocity vector of a point in a rigid body

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{v}_B &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}_A + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{\mathrm{d}\vec{r}_{B/A}}{\mathrm{d}t} \\
\vec{a}_B &= \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \vec{v}_{B,\mathrm{rot}}
\end{aligned}
$$

In this derivation we defined the angular acceleration vector of th $\vec{\alpha}$ rigid body.
The angular acceleration vector o $\vec{\alpha}$ rigid body is the time derivative of its angular velocity vector.

$$
\vec{\alpha} \equiv \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}
$$

The general vector expression for the acceleration vector of a point ***B*** in a rigid body that has translational and rotational acceleration:

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$$

This equation shows that the acceleration of a point *B* on a rigid body consists of three contributions that are shown in

1. The translational acceleration $\vec{a}_{B,\text{trans}} = \vec{a}_A$ to the acceleration of point *A*.
2. The angular acceleration $\vec{a}_{B,\text{ang}} = \vec{\alpha} \times \vec{r}_{B/A}$:o the angular acceleration vector.
3. The centripetal acceleration $\vec{a}_{B,\text{cptl}} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$e to the angular velocity vector .

Example.2

Velocity of a point on a rigid body in planar motion. A plate ABC, of an equilateral triangle geometry, is in motion in the *x-y* plane. At the instant shown in the figure, point *B* has velocity V_B = 0.3 i + 0.6 j and the plate has angular velocity ω= 2 rad/s k. Find the velocity of point A.

Solution We are given V_B and ω, and we need to find v_A, the velocity of point A on the same rigid body. We know that, Thus, to find v_A, $\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$ d $r_{A/B}$. Now,

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\mathrm{m}\hat{\imath}) = -0.2\,\mathrm{m}\hat{\imath}$$

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$

$$\begin{aligned}
\vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \quad \vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\mathrm{m}\hat{\imath}) = -0.2\,\mathrm{m}\hat{\imath} \\
&= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\mathrm{m/s} + 2\,\mathrm{rad/s}\hat{k} \times (-0.2\hat{\imath})\,\mathrm{m} \\
&= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\mathrm{m/s} - 0.4\hat{\jmath}\,\mathrm{m/s} \\
&= (0.3\hat{\imath} + 0.2\hat{\jmath})\,\mathrm{m/s}.
\end{aligned}$$

$$\boxed{\vec{v}_A = (0.3\hat{\imath} + 0.2\hat{\jmath})\,\mathrm{m/s}}$$

The coordinate representation 2D

Relative Velocity

$$\mathbf{r}_A = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

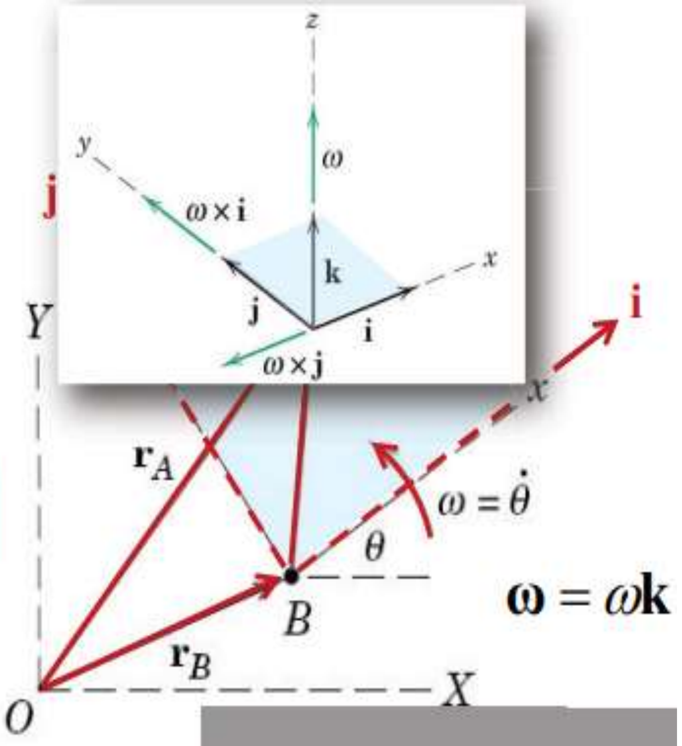
$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega}\times\mathbf{r} + \mathbf{v}_{rel}}$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \frac{d}{dt}(x\mathbf{i} + y\mathbf{j})$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}}) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + (\boldsymbol{\omega}\times x\mathbf{i} + \boldsymbol{\omega}\times y\mathbf{j}) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + (\boldsymbol{\omega}\times(x\mathbf{i} + y\mathbf{j})) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

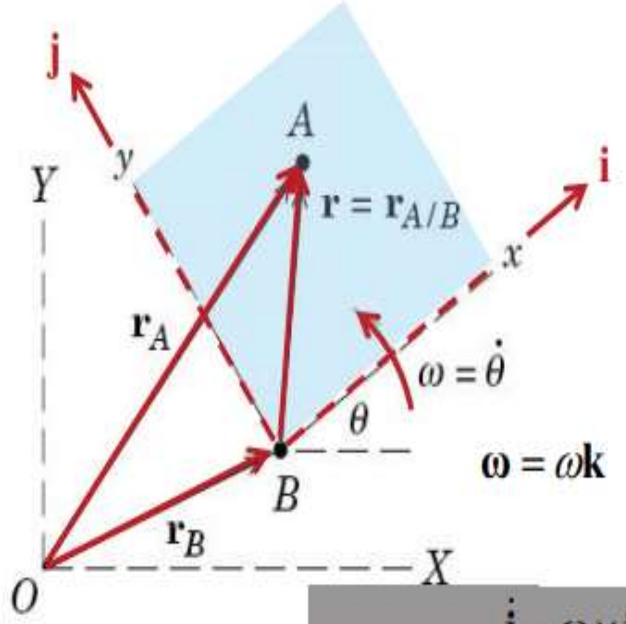


$$\boldsymbol{\omega} = \omega\mathbf{k}$$

$$\dot{\mathbf{i}} = \omega\mathbf{j} \qquad \dot{\mathbf{j}} = -\omega\mathbf{i}$$

$$\dot{\mathbf{i}} = \boldsymbol{\omega}\times\mathbf{i} \qquad \dot{\mathbf{j}} = \boldsymbol{\omega}\times\mathbf{j}$$

Relative Acceleration



$$\boldsymbol{\omega} = \omega\mathbf{k}$$

$$\mathbf{r}_A = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega}\times\mathbf{r} + \mathbf{v}_{rel}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times\dot{\mathbf{r}} + \dot{\mathbf{v}}_{rel}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r} + \mathbf{v}_{rel}) + \dot{\mathbf{v}}_{rel}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times\mathbf{v}_{rel} + \dot{\mathbf{v}}_{rel}$$

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

$$\dot{\mathbf{i}} = \boldsymbol{\omega}\times\mathbf{i}$$

$$\dot{\mathbf{j}} = \boldsymbol{\omega}\times\mathbf{j}$$

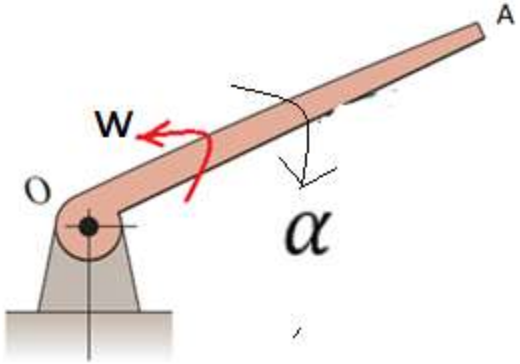
$$\dot{\mathbf{r}} = \boldsymbol{\omega}\times\mathbf{r} + \mathbf{v}_{rel}$$

$$\dot{\mathbf{v}}_{rel} = \boldsymbol{\omega}\times\mathbf{v}_{rel} + \mathbf{a}_{rel}$$

$$v = r\omega$$

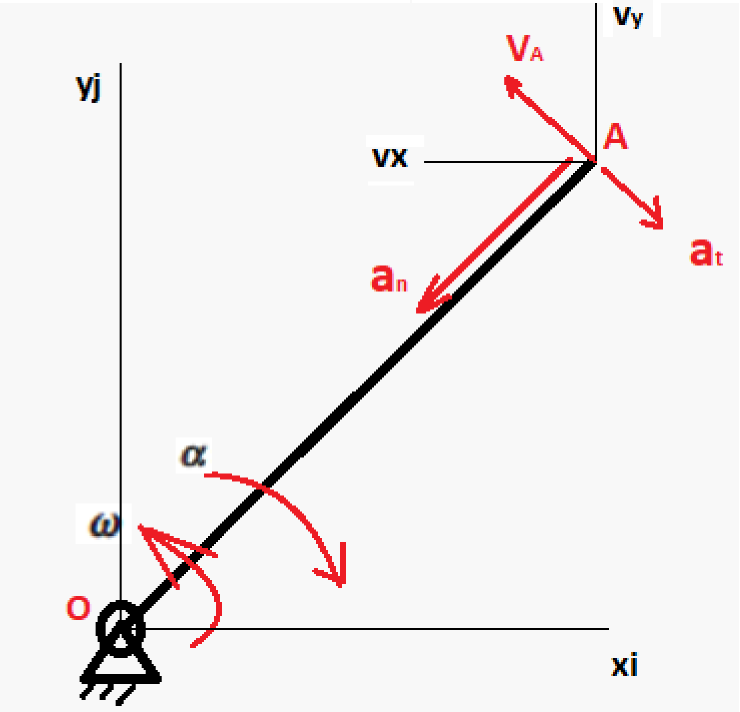
$$a_n = r\omega^2 = v^2/r = v\omega$$

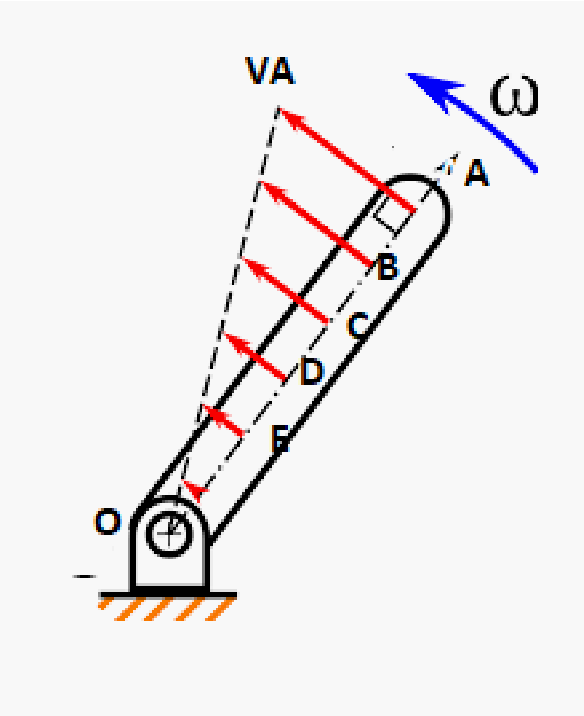
$$a_t = r\alpha$$



$$\vec{V}_A = V_x\, i + V_y\, j$$

$$\vec{a}_A = a_x\, i + a_y\, j$$





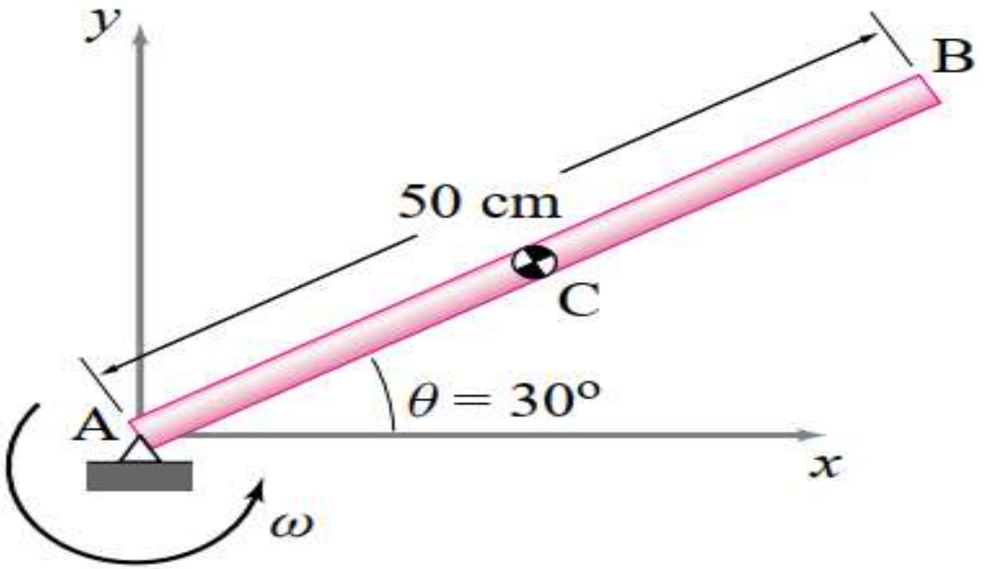
Example.2

A uniform bar AB of length l= 50 cm rotates counterclockwise about point A with constant angular speed ω. At the instant shown in Fig .3 the linear speed v_C of the center of mass C is 7:5 cm/ s.

(a) What is the angular speed of the bar?

(b) What is the angular velocity of the bar?

(c) What is the linear velocity of end B?



Relative Acceleration

Translation and Fixed Axis Rotation

A Translation

- Consider rigid body in translation:

- direction of any straight line inside the body is constant,

- all particles forming the body move in parallel lines.

- For any two particles in the body, $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$

  Differentiating with respect to time $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$

  All particles have the same velocity. $\vec{v}_B = \vec{v}_A$

- Differentiating with respect to time again,

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$

All particles have the same acceleration. $\vec{a}_B = \vec{a}_A$

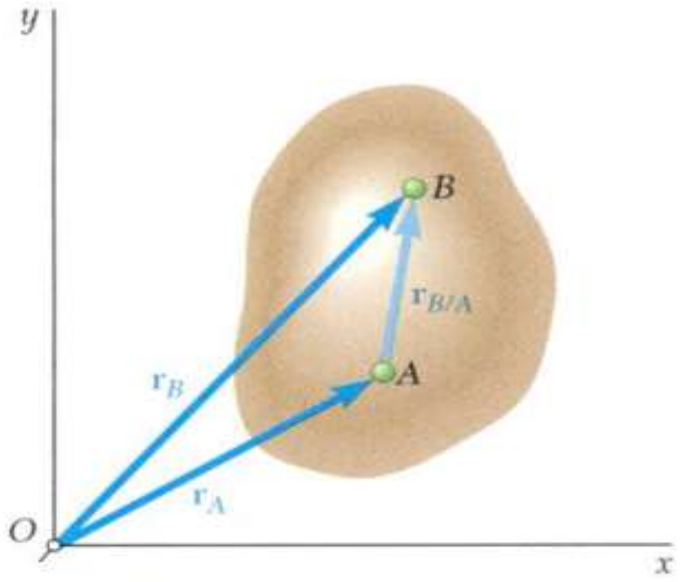


Figure .5: Translation of a rigid body.

Rotation About a Fixed Axis.

When a body rotates about a fixed axis, any point P in the body travels along a circular path.
The change in angular position, *d* , is called the angular displacement, which units of either radians or revolutions. They are related by
1 revolution = 2 π radians.
Angular velocity, ω, is obtained by taking the time derivating of angular displacement
Similarly, angular acceleration is

$$\omega = d\theta/dt \ (rad/s) \ +$$

$$\alpha = d^2\theta/dt^2 = d\omega/dt \ \text{ or } \alpha = \omega(d\omega/d\theta) + \ rad/s^2$$

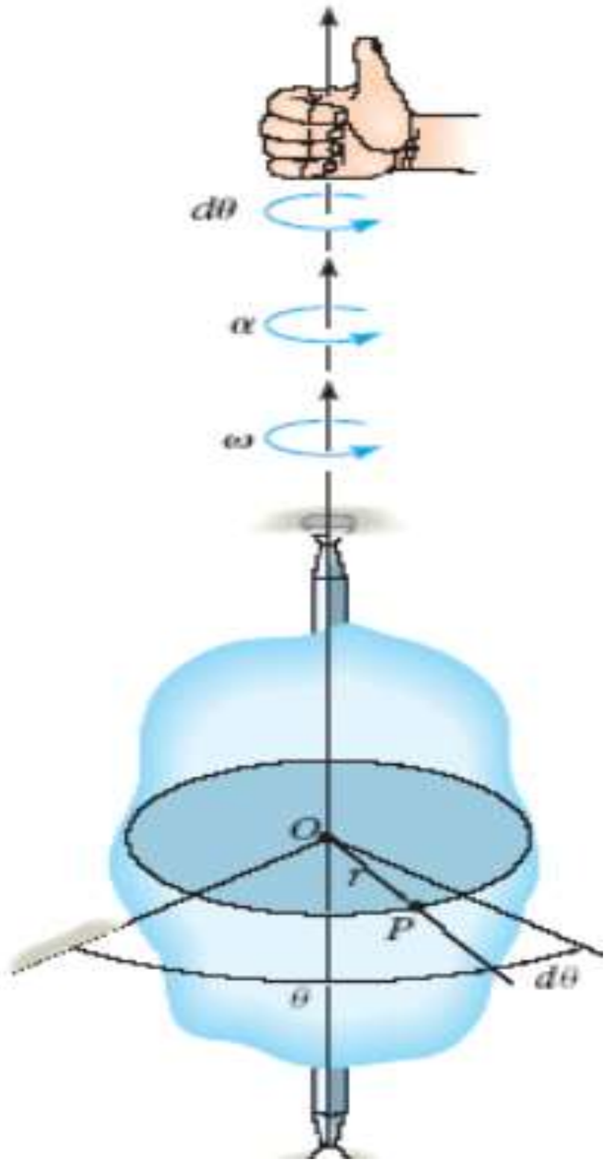


Figure .6: Rotation of a rigid body.

Velocity

Consider rotation of rigid body about a fixed axis AA

- The same result is obtained from

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \omega \vec{k} = \dot{\theta} \vec{k} = angular\ velocity$$

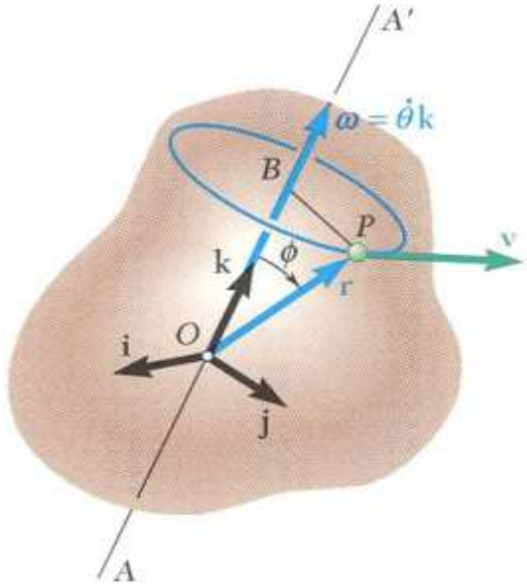


Figure .7: Rotation of a rigid body.

Acceleration

Differentiating to determine the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}$$

and

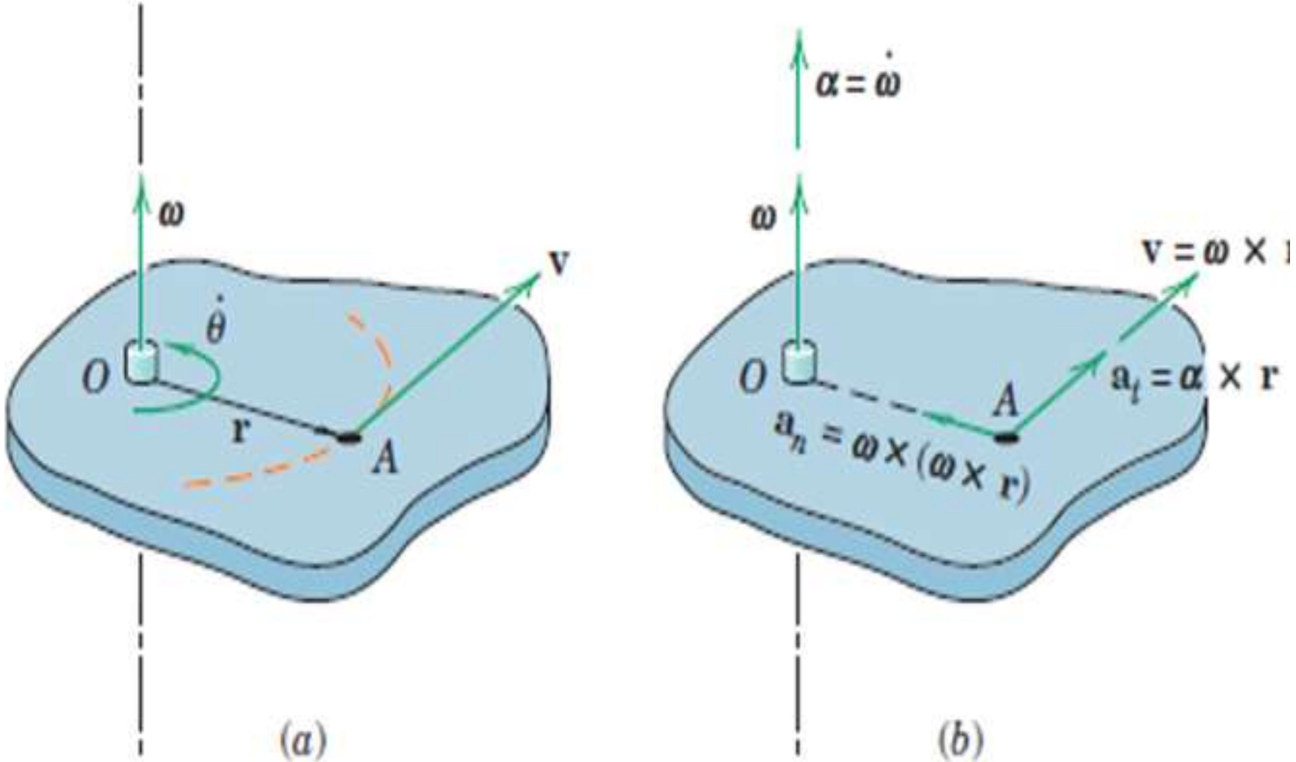
$$\frac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration$$

$$= \alpha \vec{k} = \dot{\omega} \vec{k} = \ddot{\theta} \vec{k}$$

$$\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \omega\, k \qquad \vec{r} = xi + yj$$

The order of the vectors to be crossed must be retained. The reverse order gives $\mathbf{r} \times \boldsymbol{\omega} = -\mathbf{v}$.



(a) (b)

From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

General Plane Motion

A general plane motion can always be considered as the sum of a translation and a rotation

Example 1.

Consider, for example, a wheel rolling on a straight track (Fig. 8). Over a certain interval of time, two given points A and B will have moved, respectively, from A_1 to A_2 and from B_1 to B_2. The same result could be obtained through a translation which would bring A and B into A_2 and B'_1 (the line AB remaining vertical), followed by a rotation about A bringing B into B_2.

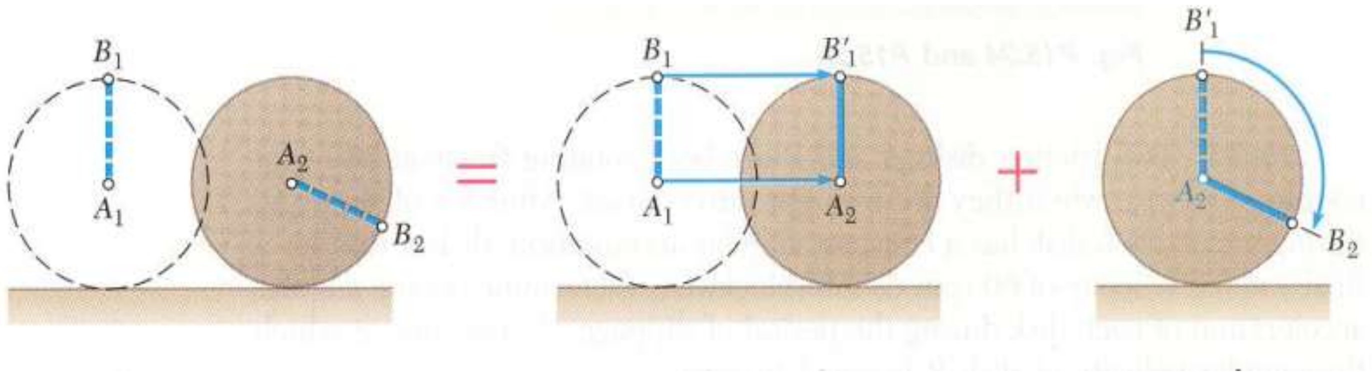


Fig.8 A General plane motion ,a combination of Translation & Rotation

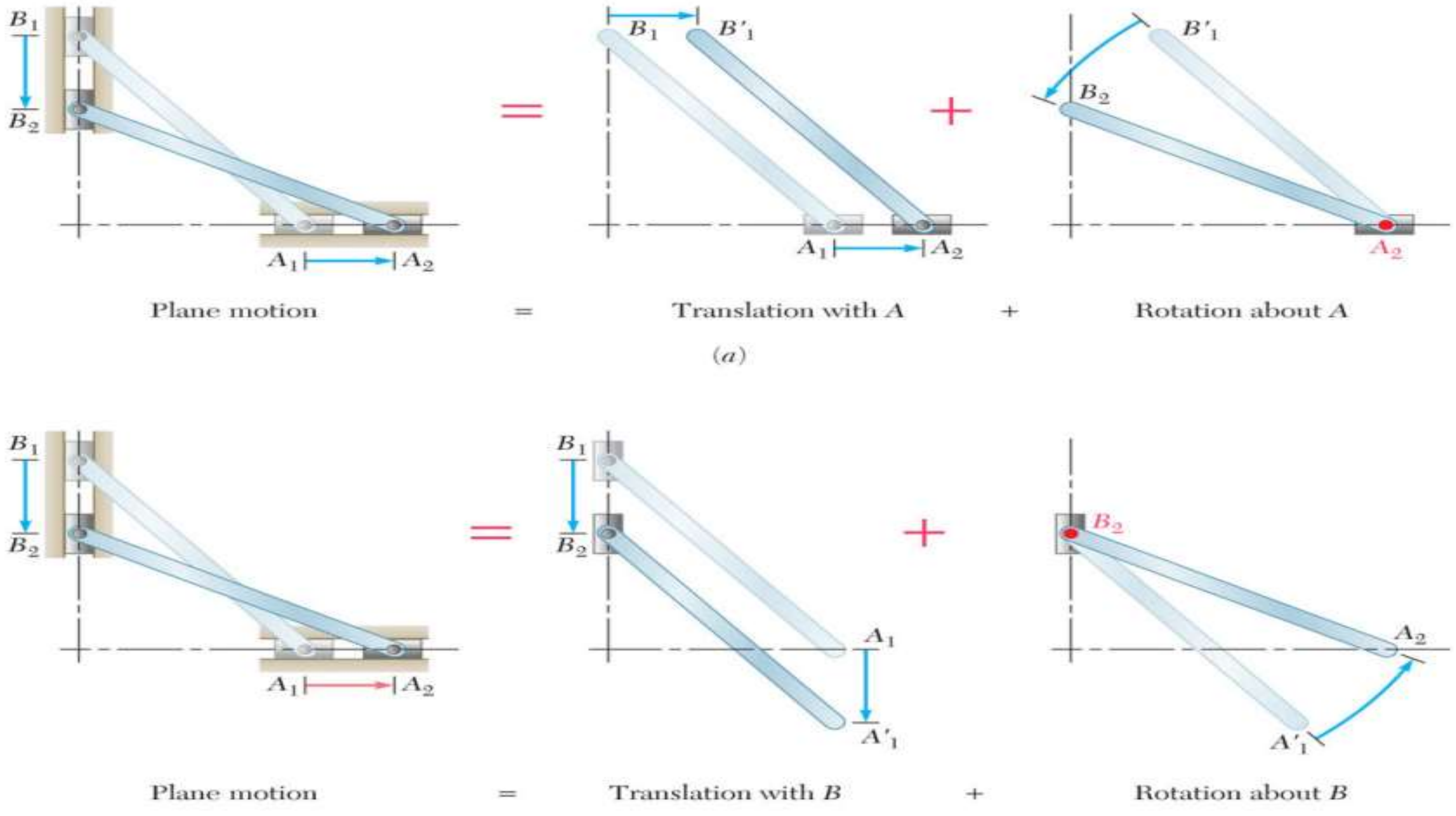


Fig.9 A General plane motion ,with sliders

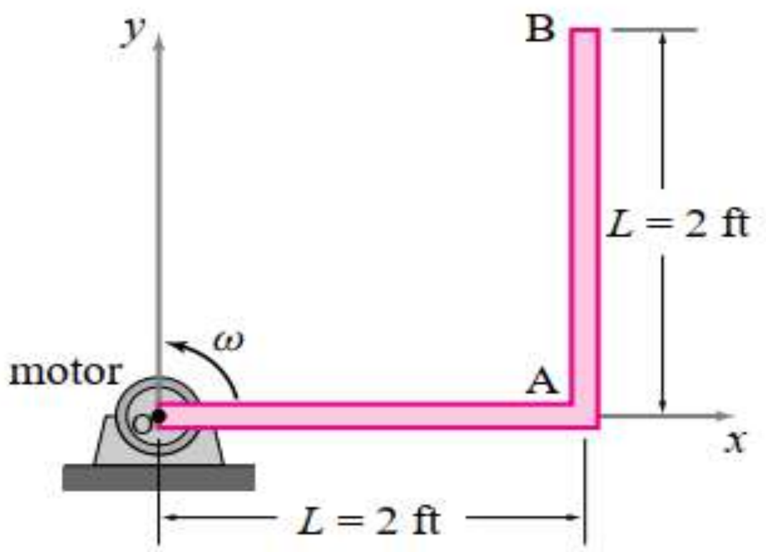
Example.3

Test the acceleration formula on something you know. Consider the 'L 'shaped bar. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 rad/s2

(i) Find the acceleration of point A.

(ii) Find the relative acceleration $\vec{a}_{B/A}$ point B with respect to point A and use the result to find the absolute acceleration of point B $(\vec{a}_B = \vec{a}_A + \vec{a}_{B/A})$.

(iii) Find the acceleration of point B directly and verify the result obtained in (ii).



Special topic in planar kinematics

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, but we will also mention more advanced topics.

2-D rolling of a round wheel on level ground

The simplest case, ***the no-slip rolling*** of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in Fig. 10. For convenience, we pick a point ***D*** which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

The arc length traversed on the wheel is the distance traveled by the wheel center.

That is

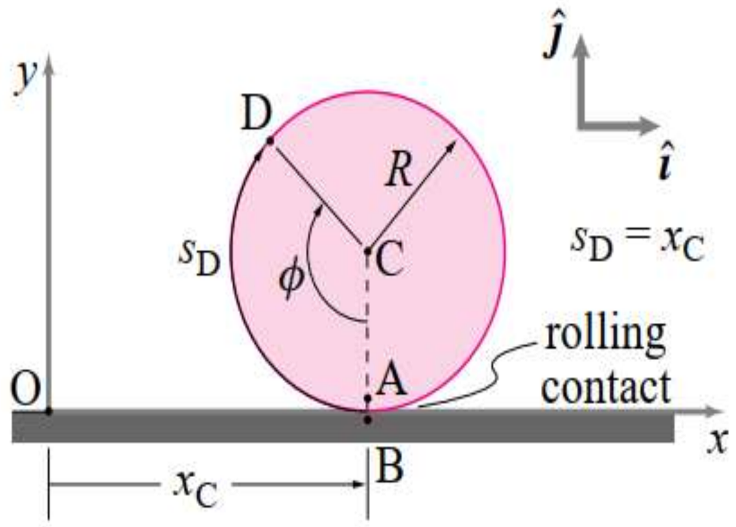
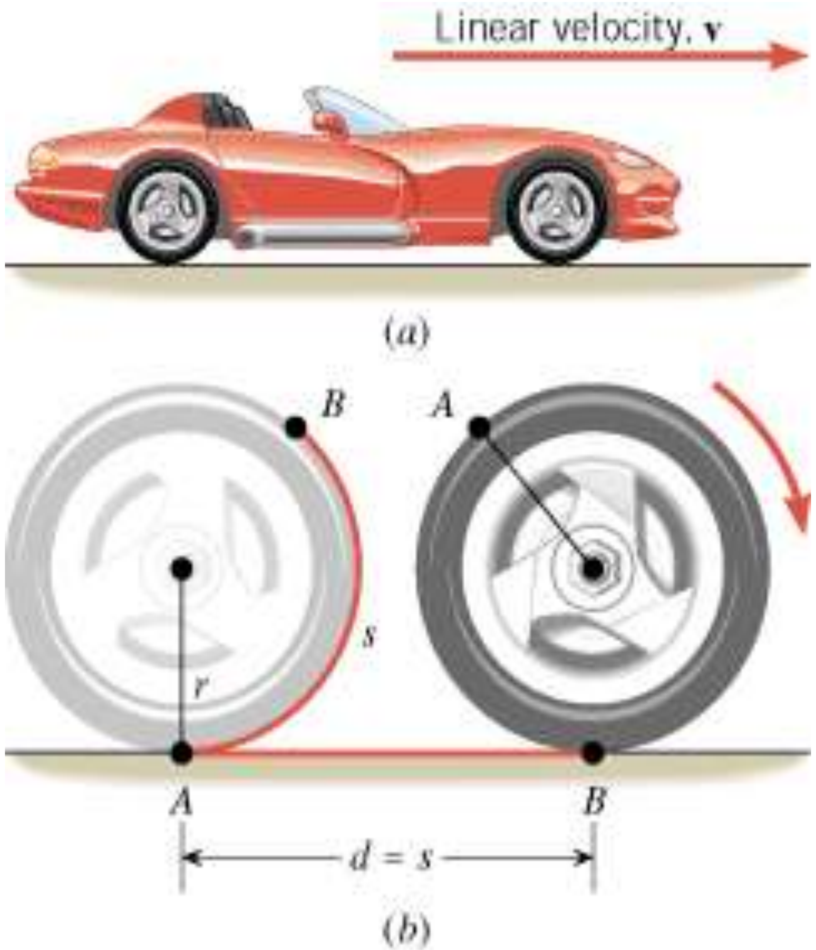


Figure .10: Pure rolling of a round wheel on a level support



$$v = r\omega$$

Rolling Motion

- (a) An automobile moves with a linear speed v.
- (b) If the tires roll and do not slip, the distance d, through which an axle moves, equals the circular arc length s along the outer edge of a tire.

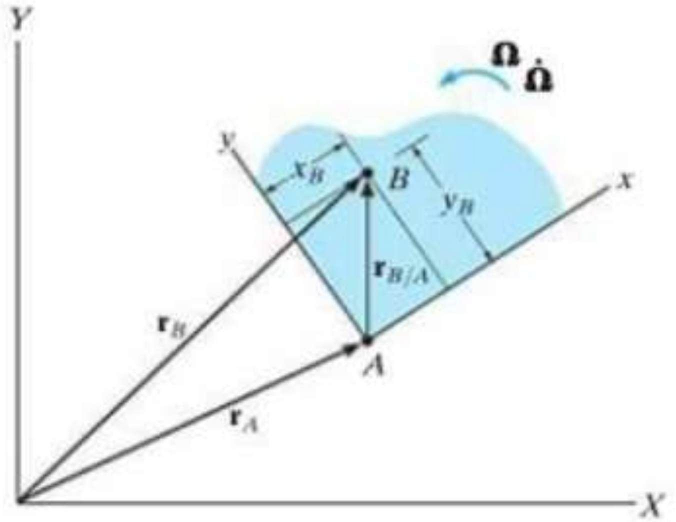
To represent the translational motion of a rigid body in space, we require three independent coordinates, namely, x, y and z, and to represent its rotational motion, we require three more coordinates, namely, angular coordinates θ_x, θ_y and θ_z. Hence, six independent coordinates are required to represent the motion of a rigid body. Thus, we say that a rigid body has six degrees of freedom.

For a system of rigid bodies, we can establish a local Cartesian coordinate system for each rigid body. Transformation matrices are used to describe the relative motion between rigid bodies.

Component movement

Position. Consider two points A and B shown in figure. Using vector addition, the three position vectors are related by the relation.

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

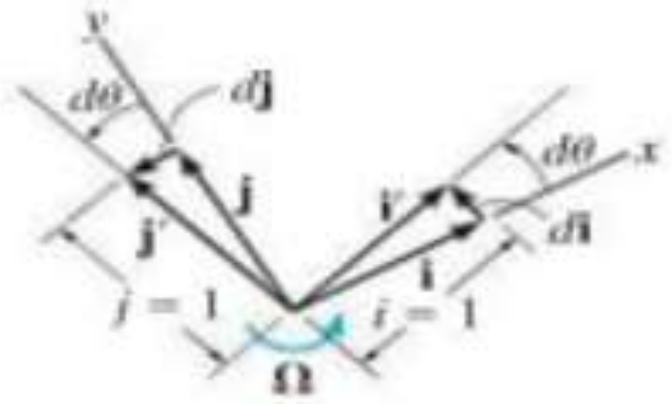


Velocity. The velocity of B is

The last term in this equation

$$\mathbf{v}_B = \mathbf{v}_A + \frac{d\mathbf{r}_{B/A}}{dt}$$

$$\begin{aligned}\frac{d\mathbf{r}_{B/A}}{dt} &= \frac{d}{dt}(x_B\mathbf{i} + y_B\mathbf{j}) \\ &= \frac{dx_B}{dt}\mathbf{i} + x_B\frac{d\mathbf{i}}{dt} + \frac{dy_B}{dt}\mathbf{j} + y_B\frac{d\mathbf{j}}{dt} \\ &= \left(\frac{dx_B}{dt}\mathbf{i} + \frac{dy_B}{dt}\mathbf{j}\right) + \left(x_B\frac{d\mathbf{i}}{dt} + y_B\frac{d\mathbf{j}}{dt}\right)\end{aligned}$$



Where

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\Omega} \times \mathbf{i} \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\Omega} \times \mathbf{j}$$

$$\frac{d\mathbf{r}_{B/A}}{dt} = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (x_B\mathbf{i} + y_B\mathbf{j}) = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A}$$

Hence the equation becomes

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

where

$\mathbf{v}_B$ = velocity of B, measured from the X, Y, Z reference

$\mathbf{v}_A$ = velocity of the origin A of the x, y, z reference, measured from the X, Y, Z reference

$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "B with respect to A," as measured by an observer attached to the rotating x, y, z reference

$\mathbf{\Omega}$ = angular velocity of the x, y, z reference, measured from the X, Y, Z reference

$\mathbf{r}_{B/A}$ = position of B with respect to A

Acceleration. The acceleration of B by driving is

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt}$$

$$\mathbf{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} = \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A}) \qquad (1$$

Finding the time derivative of $(\mathbf{v}_{B/A})_{xyz} = (v_{B/A})_x\mathbf{i} + (v_{B/A})_y\mathbf{j}$,

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = \left[\frac{d(v_{B/A})_x}{dt}\mathbf{i} + \frac{d(v_{B/A})_y}{dt}\mathbf{j}\right] + \left[(v_{B/A})_x\frac{d\mathbf{i}}{dt} + (v_{B/A})_y\frac{d\mathbf{j}}{dt}\right]$$

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = (\mathbf{a}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A}) + 2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

where

$\mathbf{a}_B$ = acceleration of B, measured from the X, Y, Z reference

$\mathbf{a}_A$ = acceleration of the origin A of the x, y, z reference, measured from the X, Y, Z reference

$(\mathbf{a}_{B/A})_{xyz}$, $(\mathbf{v}_{B/A})_{xyz}$ = acceleration and velocity of B with respect to A, as measured by an observer attached to the rotating x, y, z reference

$\dot{\mathbf{\Omega}}$, $\mathbf{\Omega}$ = angular acceleration and angular velocity of the x, y, z reference, measured from the X, Y, Z reference

$\mathbf{r}_{B/A}$ = position of B with respect to A

Example 5

Velocity and acceleration in 3-D: The rod shown in the figure rotates about the *y*-axis at angular speed 10 rad / s and accelerates at the rate of 2 rad/s2. The dimensions of the rod are $L = h = 2$ m and $r = 1$ m. There is a small mass ***P*** glued to the rod at its free end.

At the instant shown, the three segments of the rod are parallel to the three axes.

(a) Find the velocity of point ***P*** at the instant shown.

(b) Find the acceleration of point ***P*** at the instant shown