ANOVA Test

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— What is the ANOVA Test?

The ANOVA test (or Analysis of Variance) is used to compare the means of several groups. The term "ANOVA" is somewhat misleading. Although the technique's name refers to variances, the main objective of ANOVA is to study differences between means.

— What are the advantages of one-factor ANOVA?

One-factor ANOVA can help you determine whether or not significant differences exist between the groups of your independent variable.

Validity conditions of the ANOVA test

The validity conditions for the ANOVA test:

- Independence of samples.
- Normality of the distribution of measurements.
- Homogeneity of variances (equal variances).

How to perform a one-factor ANOVA test?

Single factor:

1. Choice of hypotheses:

$$H_0 : \text{ Equal means}$$

$$H_1 : \text{ At least one mean is different}$$

2. Calculation of the observed test statistic: (see below for definitions of mean squares and sums of squares)

How to perform a one-factor ANOVA? (continued)

For a single factor:

$$MS_{fa}: \text{ The between-group variance or factorial mean square, defined by: } MS_{fa} = \frac{SS_{fa}}{v_{fa}}$$

$$MS_r: \text{ The within-group variance or residual mean square, defined by: } MS_r = \frac{SS_r}{v_r}$$

How to perform a one-factor ANOVA? (continued)

Such that:

$$SS_T: \text{ The total sum of squares, defined by } SS_T = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \bar{x})^2$$

$$SS_{fa}: \text{ The factorial (between-groups) sum of squares, defined by } SS_{fa} = \sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x})^2$$

How to perform a one-factor ANOVA? (continued)

$$SS_r: \text{ The residual (within-groups) sum of squares, defined by } SS_r = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2$$

This is called the analysis of variance equation.

Next steps

3) Identification of the critical threshold:

The critical threshold is read from Table 5 (Fisher's distribution). In the table, the column gives the value of v_1 and the row gives the value of v_2.

4) Decision:

$$F_0 < F_{v_1,v_2,1-\alpha} \quad \Rightarrow \quad \text{Accept } H_0$$

$$F_0 > F_{v_1,v_2,1-\alpha} \quad \Rightarrow \quad \text{Reject } H_0$$

Example

We want to know if the addition of adjuvant substances to a vaccine modifies antibody production. For this, we measure the quantities of antibodies produced by subjects after administration of equal amounts of the vaccine, with or without an added adjuvant substance. We obtain the rates:

Groups: *Without substance* *With alum* *With phosphates*

(data presented in slides; values reproduced in solution section)

Under appropriate assumptions, does the vaccine's effectiveness depend on:

1. The presence of adjuvant substances?
2. The nature (type) of the adjuvant?

Solution (slide content)

1) One-factor ANOVA (comparison of means):

- **Substance:** Factor (qualitative independent variable) with 3 levels / samples.
- **Quantity of antibodies:** Dependent quantitative variable.

Choice of hypotheses:

$$H_0: \text{ All means are equal}$$

$$H_1: \text{ At least one of the three is different}$$

2) Calculation of the observed test statistic:

(Manual calculation table follows — the slides show a hand computation layout. Reproducing numerical summaries as on slides:)

	Without substance	With alum	With phosphates
Observations (x_{ij})	1.3, 3.0, 1.2	4.5, 4.3, 6	4.2, 3.3
k (number of groups)	3	3	2
n_i (group sizes shown in slides)	5	6	5
$\sum_j x_{ij}$	8	24	13
T (grand total shown)	45		
$\left(\sum_j x_{ij}\right)^2$	64	576	169
$\frac{\left(\sum_j x_{ij}\right)^2}{n_i}$	128	96	33.8
$\sum x_{ij}^2$	20	106	39
T (sum of squares total shown)	165		

Computation (formulas used)

$$SS_T = \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}^2$$

$$SS_{fa} = \sum_{i=1}^{k} \frac{\left(\sum_{j=1}^{n_i} x_{ij}\right)^2}{n_i}$$

$$SS_r = SS_T - SS_{fa}$$

$$F_0 = \frac{MS_{fa}}{MS_r} = \frac{SS_{fa}/v_{fa}}{SS_r/v_r}$$

3) Identification of the critical threshold:

The critical value is read from Table 5 (Fisher distribution). Use the row corresponding to v_2 and the column corresponding to v_1. If the exact value is not present in the table, one may take the midpoint between the two adjacent values (left and right).

Example shown in slides:

$$F_{2;14;0.95}$$

4) Decision:

If $F_0 > F_{v_1,v_2,1-\alpha}$, then reject H_0. In the example, F_0 exceeded the critical value, so at least one mean differs — i.e., the presence or absence of the substance influences vaccine effectiveness.

Table 5 (Fisher's distribution)

(The slides reference Table 5 — Fisher distribution table — which is not reproduced here.)

ANOVA for comparison of two means (single factor with 2 levels)

Problem setup:

- Factor: added substance with 2 levels (with alum; with phosphates).
- Response: quantity of antibodies (quantitative).

Hypotheses:

$$H_0: \text{ Means equal}$$

$$H_1: \text{ Means different}$$

Calculation:

(Manual computation table reproduced in slides for the two-group case:)

	With alum	With phosphates
Observations	4.5, 4.3, 6	4.2, 3.3
k	2	
n_i	6	5
$\sum_j x_{ij}$	24	13
$\left(\sum_j x_{ij}\right)^2$	576	169
$\frac{\left(\sum_j x_{ij}\right)^2}{n_i}$	96	33.8
$\sum x_{ij}^2$	106	39
Total sums	145	

Identification of critical threshold:

Using Table 5 (Fisher distribution), row corresponds to v_2 and column to v_1. Example shown:

$$F_{1;9;0.95}$$

Decision:

If $F_0 < F_{1;9;0.95}$ then accept H_0. In the slides, the decision was to accept H_0 for this two-level comparison, meaning the nature (type) of the adjuvant did not influence vaccine efficacy in that test.

Remarks

1. The ANOVA test is a one-sided test.
2. If the homogeneity of variances condition (equal variances) is not satisfied as stated, you should always begin with a test for comparison of variances.

Slide: Remarks (continued)

1. The significance level indicates:
   - "Significant difference" means (as usual) $p < \alpha$.
   - "Highly significant difference" indicates a still smaller p-value (more strict).

Presentation of the variance comparison test

We propose to compare the variances of a population P_1 and a population P_2 using the sample variances from independent random samples drawn from each population. The samples are independent.

How to perform a variance comparison test?

Several tests are used:

1. **Bartlett's test:** applies to multiple samples of unequal sizes. Two validity conditions: normality of distribution and independence of samples.
2. **Levene's test:** commonly used in SPSS.
3. **Approximate method:** faster to perform than the first two. Two validity conditions: normality of distribution and independence of samples. This approximate method consists of comparing the two extreme variances.
4. **Fourth method:** consists of pairwise comparison of variances two by two.

How to perform the variance comparison test? (continued)

1. Choice of hypotheses:
$$H_0 : \ \sigma_1^2 = \sigma_2^2 \quad \text{or} \quad H_1 : \ \sigma_1^2 \neq \sigma_2^2$$
or for one-tailed:
$$H_0 : \ \sigma_1^2 \leq \sigma_2^2 \quad \text{vs} \quad H_1 : \ \sigma_1^2 > \sigma_2^2$$
2. Calculation of the observed test statistic: Use the ratio of sample variances; always place the larger sample variance in the numerator.

Identification of critical threshold for F

For F_{v_1,v_2} such that v_1 is the numerator degrees of freedom and v_2 is the denominator degrees of freedom, we decide:

$$\text{If } F_0 < F_{v_1,v_2,1-\alpha} \quad \text{Accept } H_0$$

$$\text{If } F_0 > F_{v_1,v_2,1-\alpha} \quad \text{Reject } H_0$$

(Refer to Table 5(A) and Table 5(B) in the slides.)

What is the Fisher–Snedecor distribution?

Let two independent χ^2 distributions be:

$$\chi^2_{\nu_1} \quad \text{and} \quad \chi^2_{\nu_2}$$

with ν_1 and ν_2 degrees of freedom, respectively. Denote by F_{ν_1,ν_2} the quotient:

$$F_{\nu_1,\nu_2} = \frac{(\chi^2_{\nu_1}/\nu_1)}{(\chi^2_{\nu_2}/\nu_2)}$$

Remark:

$$F_{\nu_1,\nu_2} = \frac{1}{F_{\nu_2,\nu_1}}$$

— Comparing precision of two assay methods

We want to compare the precision of two methods for assaying menthol in peppermint oil. For this, we measure menthol in 16 vials by both methods. The variances of the results obtained are respectively 0.013 (method 1) and 0.024 (method 2).

Questions:

1. At the 5% risk level, can we say that these two methods do not have the same precision? (Assumptions of validity are satisfied.)
2. At the 5% risk level, can we say that method 1 is more precise than method 2? (Assumptions of validity are satisfied.)

Solution

1. Choice of hypotheses:

$$H_0: \ \sigma_1^2 = \sigma_2^2$$

$$H_1: \ \sigma_1^2 \neq \sigma_2^2$$

2. Calculation of the observed test statistic:

$$F_0 = \frac{s_1^2}{s_2^2}$$

(Always put the larger variance in the numerator.)

Identification of critical threshold (example)

For the example:

$$F_{v_1,v_2} = 2.86 \quad \text{(value shown in slides)}$$

Decision: $F_0 < F_{v_1,v_2}$ so accept H_0. Conclusion: We cannot say the precisions of the two methods are different.

Second question (one-sided)

1. Choice of hypotheses for one-sided:

$$H_0: \ \sigma_1^2 \leq \sigma_2^2$$

$$H_1: \ \sigma_1^2 > \sigma_2^2$$

2. Calculation of F_0 (again, larger variance in numerator).

Identification of critical threshold (one-sided)

In the slides the example gives:

$$F_{v_1,v_2} = 2.40$$

Decision: $F_0 < F_{v_1,v_2}$ so accept H_0. Conclusion: We cannot say method 1 is more precise than method 2.

Remarks summary

1. The ANOVA test is unilateral (one-sided).
2. If the homogeneity of variances condition is not satisfied, begin with a variance comparison test.
3. The significance level interpretation:
   - "Significant difference" means $p < \alpha$.
   - "Highly significant difference" means a much smaller p-value.

End of translated slides (Chapitre06.pdf).