Poiseuille flow

les hypothèses typiques de l'écoulement de Poiseuille: laminaire, établi, fluide incompressible

Ces conditions sont nécessaires pour que la **loi de Poiseuille** s'applique (débit, profil de vitesse, relation pression-viscosité, etc.).

1- Laminaire

Un écoulement « laminaire » signifie que les couches du fluide glissent les unes sur les autres sans mélange turbulent.

- Si une de ces hypothèses n'est pas vérifiée (par exemple un fluide compressible, un écoulement turbulent, ou un régime non-stationnaire), la loi de Poiseuille ne sera plus rigoureusement applicable, ou il faudra prendre en compte des corrections.

2- Établi (ou stationnaire)

- « Établi » veut dire que les propriétés du fluide (vitesse, pression…) ne varient pas dans le temps à un point donné : l'écoulement est **permanent**. En régime établi, on peut faire des simplifications dans les équations de Navier-Stokes (par exemple, la dérivée temporelle disparaît).
- Cette hypothèse est essentielle pour la loi de Poiseuille : elle garantit que le débit est constant dans le temps.

Donc l'« établi » est bien une hypothèse classique.

3. Fluide incompressible

•L'incompressibilité signifie que la masse volumique **ρ** du fluide reste (pratiquement) constante pendant l'écoulement.

Beaucoup de liquides (comme l'eau) sont très bien modélisés comme incompressibles dans ces conditions.

•La loi de Poiseuille (Hagen-Poiseuille) suppose effectivement un fluide incompressible.

à partir des équations de Navier–Stokes) qui montre comment les hypothèses « laminaire, établi, incompressible » mènent à la loi de Poiseuille pour un tube circulaire droit.

Hypothèses et configuration

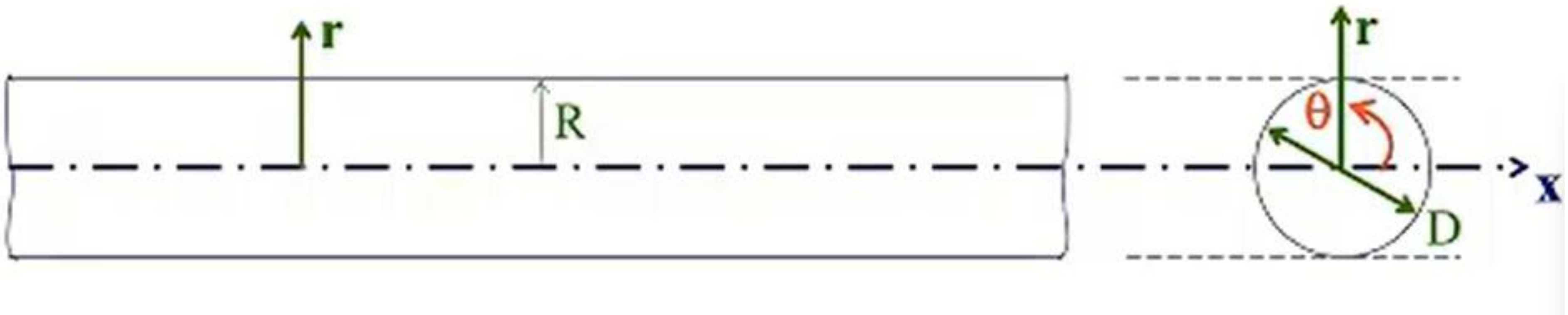
•Écoulement dans un tube droit de rayon Ret longueur L.

•Coordonnées cylindriques (r, θ, z)avec $z$axe axial.

•Fluide newtonien de viscosité dynamique μ et incompressible.

•Écoulement **laminaire**, **axial**, **axisymétrique** (pas de dépendance en θ), et **établi** (stationnaire) — donc la vitesse n'a que la composante axiale $u_z(r)$,notée $u(r)$,et $\partial/\partial t = 0$.

•Pas de forces de volume (gravité négligée), conditions aux limites : $u(R) = 0$)no-slip), $u$finie en $r = 0$



The velocity field of a fully developed flow:

•independent of time

•independent of the axial coordinate (identical in every straight section).

Established:
•independent of time
•independent of the axial coordinate

$$\begin{cases} (\partial V_i / \partial t) \equiv 0 \\ (\partial V_i / \partial x) \equiv 0 \end{cases}$$

In flows through cylindrical pipes, the velocity field is parabolic (paraboloid profile), which gives us symmetry with respect to the angle θ; this is what we call circular rotational symmetry. This symmetry is valid for both velocity and pressure.

Revolution symmetry:

$$\begin{cases} (\partial V_i / \partial \theta) \equiv 0 \\ (\partial p^* / \partial \theta) \equiv 0 \end{cases}$$

Let us use cylindrical coordinates (O, r, θ, x) to write the Navier-Stokes equations and the Continuity equation.

Continuity equation

$$\frac{1}{r}\frac{\partial (r V_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_x}{\partial x} = 0$$

Momentum equations

$$\begin{cases} \rho\left[\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_x \frac{\partial V_r}{\partial x} - \frac{V_\theta^2}{r}\right] = -\frac{\partial p^*}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right] + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_r}{\partial x^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right\} \\ \rho\left[\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_x \frac{\partial V_\theta}{\partial x} + \frac{V_r V_\theta}{r}\right] = -\frac{1}{r}\frac{\partial p^*}{\partial \theta} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta)\right] + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial x^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta}\right\} \\ \rho\left[\frac{\partial V_x}{\partial t} + V_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_x}{\partial \theta} + V_x \frac{\partial V_x}{\partial x}\right] = -\frac{\partial p^*}{\partial x} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_x}{\partial \theta^2} + \frac{\partial^2 V_x}{\partial x^2}\right\} \end{cases}$$

Hypotheses of the study (Poiseuille flow):

1.Steady flow:

$$\begin{cases} \frac{\partial V_i}{\partial t} \equiv 0 \\ \frac{\partial V_i}{\partial x} \equiv 0 \end{cases} \qquad i = (r, \theta, x)$$

2.Revolution symmetry:

$$\frac{\partial V_i}{\partial \theta} \equiv 0 \qquad \frac{\partial p^*}{\partial \theta} \equiv 0$$

According to the hypotheses, the velocity depends only on r, so it varies only with r.

$\Rightarrow$ **One-dimensional flow** $\boldsymbol{V_i = V_i(r)}$

According to the Continuity Equation, the radial component of the velocity field is identically zero.

$$\frac{1}{r}\frac{\partial (r V_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_x}{\partial x} = 0 \longrightarrow [r V_r(r)] = Cte$$

n.p.c.: non-penetration condition

C.L. : en $r = R$, on a : $V_r(r = R) = 0$ (c.n.p) $\longrightarrow$ $R\,V_r(R) = 0$

$\left[r\,V_r(r)\right] = Cte$

$R\,V_r(R) = 0$

$Cte = 0$ $\longrightarrow$ $V_r \equiv 0$

Continuity equation $\Rightarrow$ $\boldsymbol{V_r = 0}$

Momentum equations

$$\begin{cases} -\dfrac{\rho V_\theta^2}{r} = -\dfrac{\partial p^*}{\partial r} & \text{Projection radiale (suivant } r) \\ 0 = \mu \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}(r V_\theta)\right] & \text{Projection tangentielle (sur } \theta) \\ 0 = -\dfrac{\partial p^*}{\partial x} + \mu \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial V_x}{\partial r}\right) & \text{Projection axiale (suivant } x) \end{cases}$$

So the general properties of Poiseuille flow are:

Steady flow:

$$\begin{cases} (\partial V_i / \partial t) \equiv 0 \\ (\partial V_i / \partial x) \equiv 0 \end{cases}$$

Revolution symmetry:

$$(\partial V_i / \partial \theta) \equiv 0$$

$$(\partial p^* / \partial \theta) \equiv 0$$

$V_i = V_i(r)$

One-dimensional flow

$V_r \equiv 0$

$V_\theta \equiv 0$

Uni-directional flow

Projection radiale (suivant r)

Projection tangentielle (sur θ)

Projection axiale (suivant x)

$$\begin{cases} 0 = -\dfrac{\partial p^*}{\partial r} \\ 0 = -\dfrac{\partial p^*}{\partial \theta} \\ 0 = -\dfrac{dp^*}{dx} + \mu \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dV_x}{dr}\right) \end{cases}$$

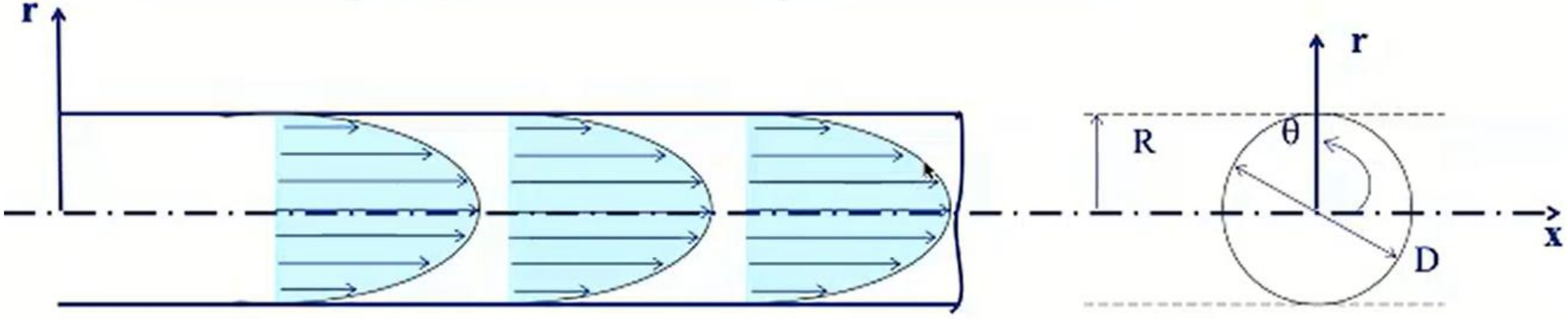
The first two equations give $\mathbf{p^* = p^*(x)}$.

The pressure is uniform over the entire cross-section, i.e., it varies only along the x-direction.

Finally, here is the velocity profile in a straight cylindrical pipe, far from the inlet, when the fluid is incompressible, Newtonian, isothermal, and flows in a fully developed regime.

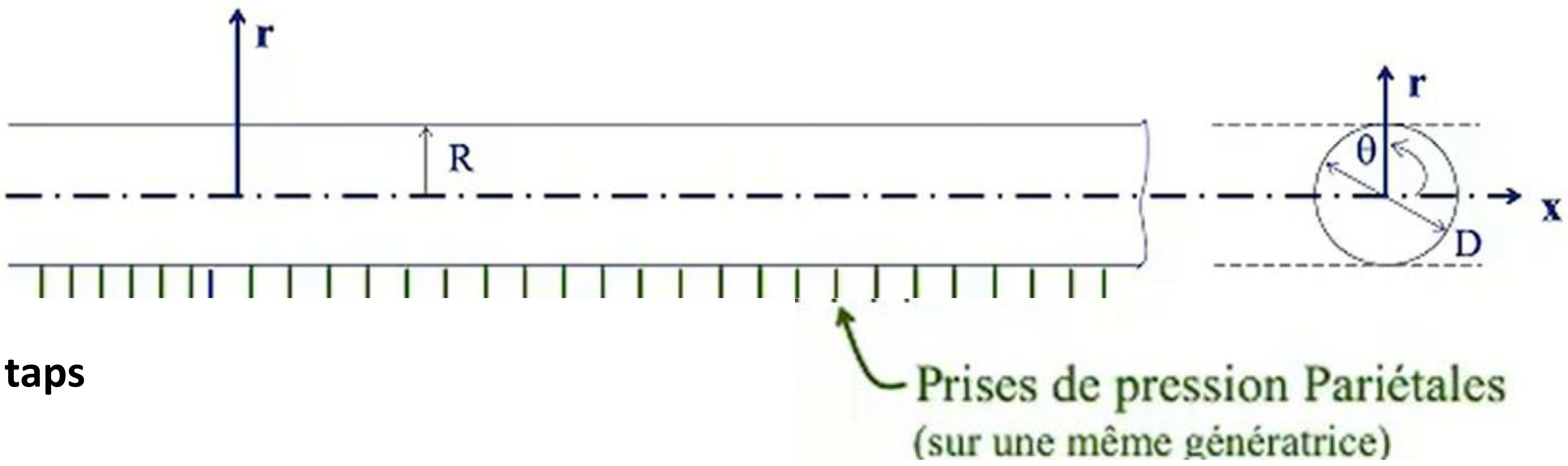
$$V_x(r) = \frac{1}{4\mu}\frac{dp^*}{dx}\left(r^2 - R^2\right)$$

The profile is parabolic far from the inlet because the flow is fully developed ; therefore it is the same parabola since the velocity is constant along x $\left(\frac{dV_i}{dx} = \mathbf{0}\right)$.



How can we experimentally demonstrate that the flow is established (or fully developed)?

- This means proving that **the velocity profile no longer changes in the direction of flow**. One of the simplest and most common methods **is indeed to use wall pressure taps**.



Experimental method: wall pressure taps

Principle:

Several pressure taps are installed along the wall of a conduit (for example, a pipe) at regular distances from the inlet, and the static pressure $P(x)$ is measured at these points.

Expected observation:

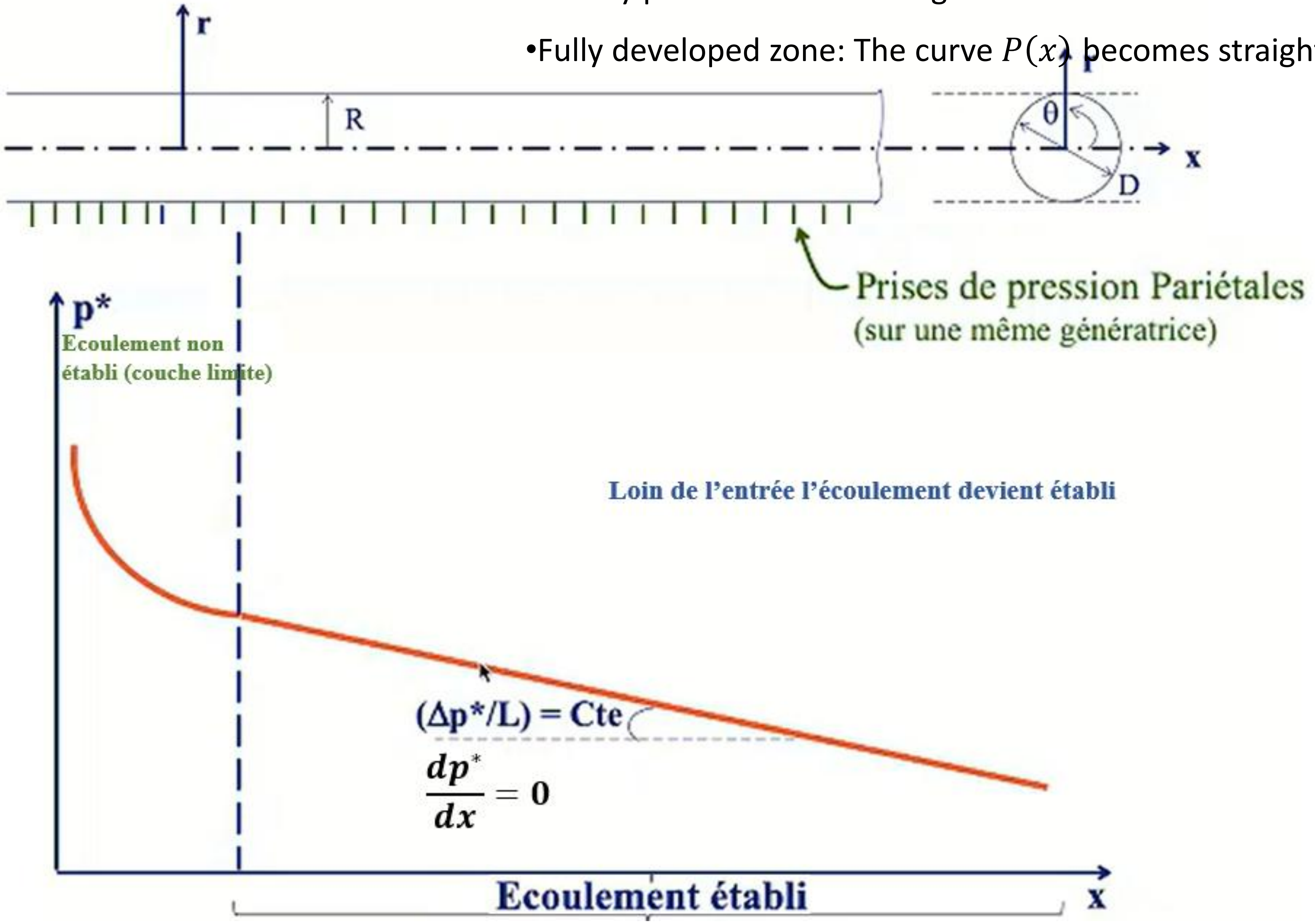
•In the steady regime, the head loss (pressure drop) becomes linear with the distance $\boldsymbol{x}$.

•In other words, the slope $\frac{dP}{dx}$ becomes constant.

•This linearity means that **the velocity profile no longer changes**, because resistance to movement (viscosity) is balanced by a constant pressure gradient.tim

How to interpret the measurements:

- Entry zone (not fully developed): The pressure drop varies in a non-linear way because the velocity profile is still evolving.
- Fully developed zone: The curve $P(x)$ becomes straight, indicating fully developed flow.



🧪 Typical experimental illustration:

1. Use a differential manometer between two successive pressure taps.
2. Record ΔP (pressure difference) between these taps at different positions.
3. Plot ΔP as a function of the distance between the taps.
4. Identify the region where ΔP becomes proportional to the distance—this is the fully developed zone.

📌 Remark:

It is also possible to experimentally validate fully developed flow by measuring the velocity profile. The main methods cited are:

- Using a Pitot tube (noted as more delicate).
- Using anemometry (for example, hot-wire anemometry), although these techniques are more complex and costly.