Chapiter 02

Concepts of Transfer Phenomena

Introduction to transfer phenomena

A transfer phenomenon (also known as a transport phenomenon) is an irreversible process during which a physical quantity is transported by means of molecules and originates from the inhomogeneity of an intensive quantity. It is the spontaneous tendency of physical and chemical systems to make these quantities uniform that causes transport.

The study of each transport phenomenon refers to a certain entity (characteristic) being transferred, for example: the amount of momentum required to increase the speed of a fluid, the heat needed to vaporize a liquid, and the mass of liquid being transported through a pipe or the dispersion of a colored liquid within another transparent liquid.

This rendering is consistent with accepted terminology in physics and engineering for describing transport phenomena and the main quantities involved.

The bodies that ensure the transfer of these physical quantities are called charge carriers.

A) Heat (thermal) transfer:

Pour lequel la grandeur transférée est la chaleur (Température), ce transfert s'effectue entre deux zones où règnent des températures différentes : il se fait toujours de la température la plus élevée vers la température la plus basse (moins élevée). La différence de température est appelée : **la force motrice** du transfert thermique.

B) Mass (matter) transfer:

When the quantity being transferred is matter (mass concentration), this transfer occurs between two regions with different mass concentrations; it always proceeds from the higher concentration to the lower concentration. The difference in concentration is called the **driving force** of mass transfer.

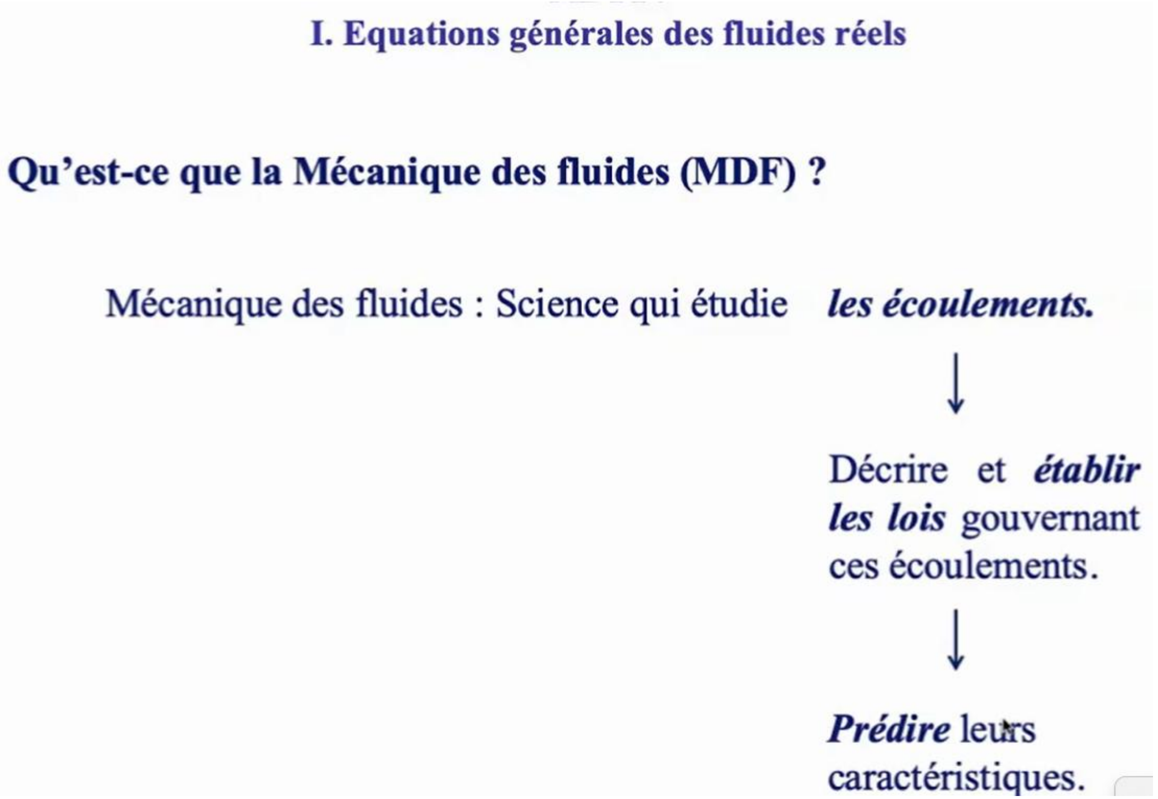
c) Transfer of momentum

When the quantity being transferred is momentum (velocity), this transfer occurs between two entities that have different velocities; it always proceeds from the entity with the higher velocity to the one with the lower velocity. The difference in velocity is called the **driving force** for momentum transfer.

Mechanics was the study of bodies in motion. However, for a liquid or a gas, we do not say that it has motion; we say that it flows. A liquid and a gas flow. It is a displacement of matter in space, but it is not strictly speaking a movement. It would be more accurate to call it a flow.

So, the fluid will flow in this manner. It will have a certain velocity, a certain flow rate, and a certain pressure. We will say that if these conditions are met, this behavior will occur.

Fluid mechanics was used to predict and describe flows. In order to predict the flow, I need equations.



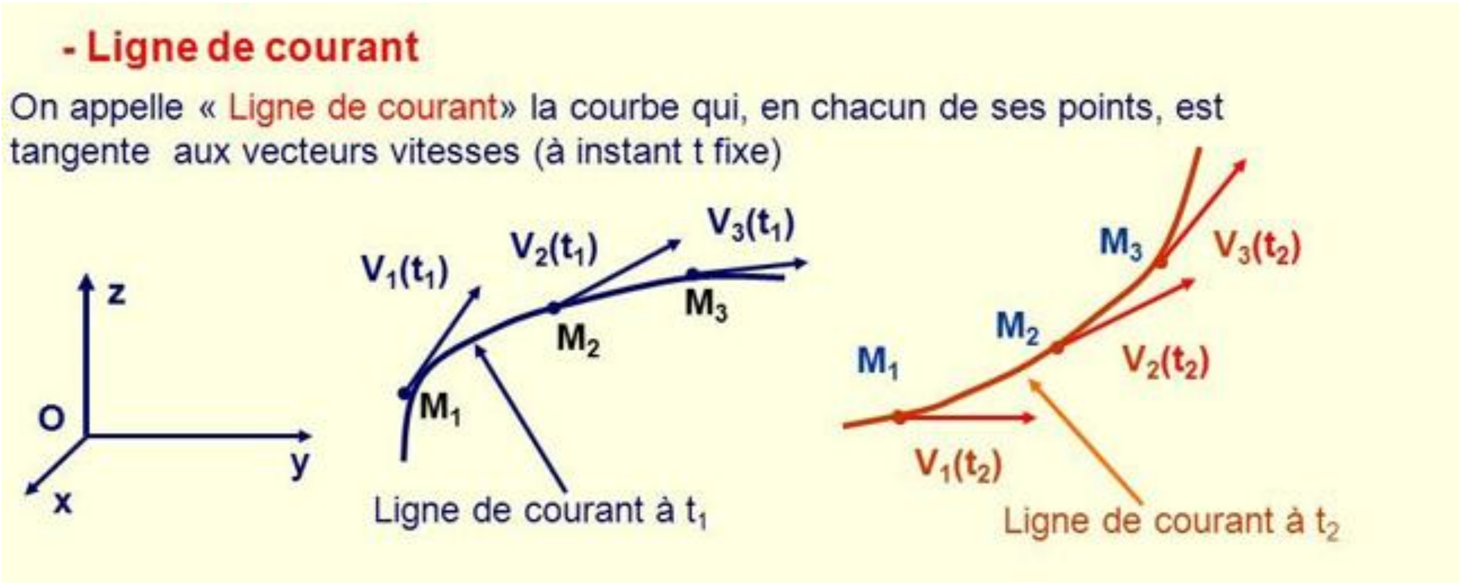
So, what are the equations that govern, that predict the flows?

The study of fluid particle motion constitutes the study of the fluid's motion itself, independently of its causes and the laws that govern it (without considering the forces that give rise to it). Therefore, only the relationships between the positions of the liquid particles and time are considered.

II.1- Definition

Streamlines

A streamline is a curve in space that describes the motion of a fluid. It corresponds to a line of the velocity field of the flow at a given instant within a control volume. It is a line along which the velocity vector is tangent at every point, as shown in the following figure.

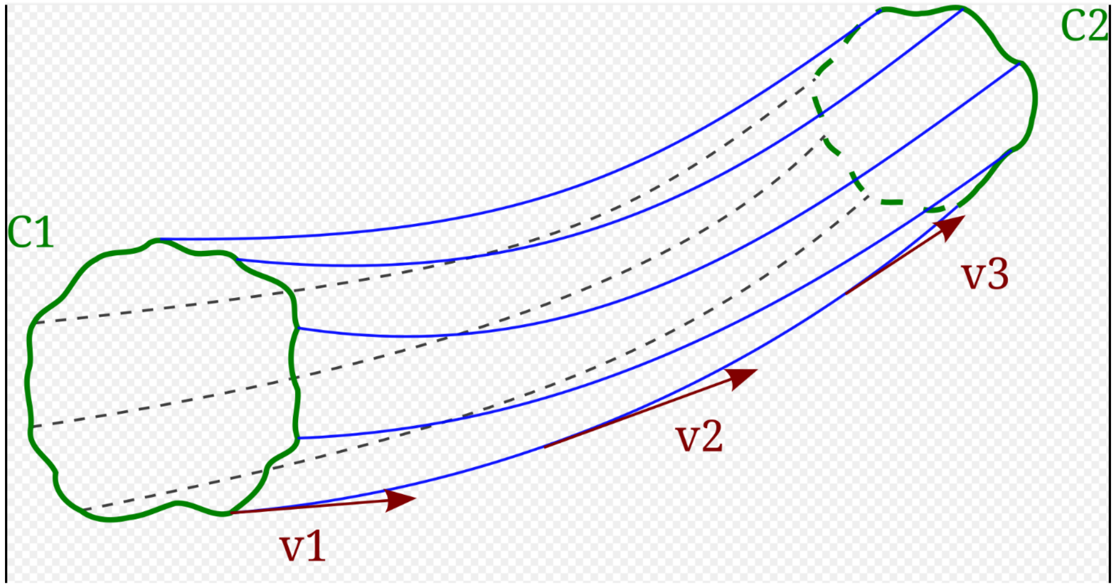


From a mathematical point of view, if an elementary displacement $\boldsymbol{dl} = \boldsymbol{dl_i e_i}$ is considered along a streamline, it will be defined by the following relation: $\vec{\boldsymbol{u}} \wedge \overrightarrow{\boldsymbol{dl}} = \vec{\boldsymbol{0}}$ (the two vectors are parallel).

This implies, in the Cartesian coordinate system, the following relation:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$ ***(*** **The mathematical definition of a streamline*****)***

A **stream tube** is a set of streamlines based on a closed contour placed inside the flow.



Trajectory: This is the curve described over time by a fluid particle, i.e., each fluid particle followed in its motion (Lagrangian description) describes a trajectory, which is determined by solving the following equation:

We have: $\frac{dx}{dt} = u(t, x, y, z)$; $\frac{dy}{dt} = v(t, x, y, z)$; $\frac{dz}{dt} = w(t, x, y, z)$

So: $dt = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

In **steady state**, the trajectories of fluid particles and the streamlines coincide. They form a network **of curves that remains constant over time**. However, in **unsteady state**, streamlines change over time and do not coincide with each other or with the trajectories of the particles; while the trajectories themselves do not intrinsically depend on them.

Acceleration of a fluid particle

Consider a fluid particle in flow at time t; the velocity field is given by:

$$\vec{V} = \vec{V}(x, y, z, t).$$

La dérivée particulaire (ou dérivée matérielle) de la vitesse donne l’accélération :

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\vec{V}(x, y, z, t)$$

En développant cette expression :

$$\vec{a} = \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt}$$

Or : : $\frac{dx}{dt} = u$; $\frac{dy}{dt} = v$; $\frac{dz}{dt} = w$; $\frac{dt}{dt} = 1$; $d’où$

$$\vec{a} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

Enfin, l’accélération peut s’exprimer par ses composantes cartésiennes :

$$\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\vec{V}(x, y, z, t)$$

Finally, acceleration can be expressed by its Cartesian components:

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$\vec{a} = \frac{\partial\vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{V}}{\partial z}\frac{dz}{dt} + \frac{\partial\vec{V}}{\partial t}\frac{dt}{dt}\ ;\ \vec{a} = u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z} + \frac{\partial\vec{V}}{\partial t}$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = \frac{Du}{Dt}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = \frac{Dv}{Dt}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = \frac{Dw}{Dt}$$

local acceleration $\frac{\partial\vec{V}}{\partial t}$) temporal variation of the field at a fixed point)

convective acceleration $u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$) due to the movement of the particle in a non-uniform velocity field).

$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t}$: local acceleration (time-dependent)

$$\vec{a} = \frac{D\vec{V}}{Dt} = (\vec{V}.\vec{\nabla}).\vec{V} + \frac{\partial \vec{V}}{\partial t} \Leftrightarrow \vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} grad \vec{V}$$

$$\boldsymbol{\vec{a} = \overrightarrow{a_{locale}} + \overrightarrow{a_{convective}}}$$

Local acceleration: This is defined as the rate of increase in speed in relation to time $\boldsymbol{\frac{\partial \vec{V}}{\partial t}}$

Convective acceleration: This is the rate of increase in velocity due to a change in the position of fluid particles.

$\boldsymbol{\vec{V} grad \vec{V}}$

Special case:

If the flow is steady $\Rightarrow \boldsymbol{\frac{\partial \vec{V}}{\partial t}} = 0$

Si l'écoulement est permanent $\Rightarrow \boldsymbol{\frac{\partial \vec{V}}{\partial t}} = 0 \Rightarrow \boldsymbol{\vec{V} grad \vec{V} = 0}$.

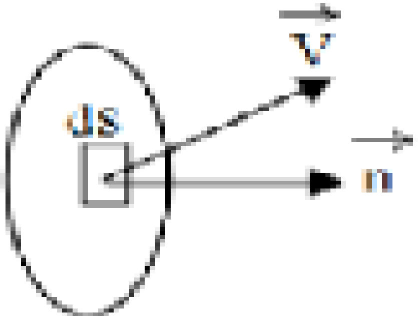
II.2 Conservation of mass

II.2.1 Mass flow and volume flow

- **Mass flow**

The mass flow rate $\boldsymbol{Q_m}$ is the amount of fluid mass passing through a given surface per unit time. It is defined by the relation :

$$Q_m = \frac{dm}{dt} = \int \rho\vec{V}\, ds\, \vec{n}$$



The mass flow rate Q_m *is expressed in* (Kg/s).

Where:

$\boldsymbol{\rho}$ is the fluid density,

$\vec{\boldsymbol{V}}$ is the velocity vector,

$\vec{\boldsymbol{n}}$ is the unit normal to the surface element **dS**.

- **Volume flow**

The volume flow rate Q_V is defined as the volume of fluid passing through a given section per unit time. It is expressed as :

$$\boldsymbol{Q_V = \frac{dV}{dt} = \rho \frac{dm}{dt} = \int \vec{V}\, ds\, \vec{n}}$$

$where\ Q_V\ \ is\ expressed\ in\ (m^3/s\,)$

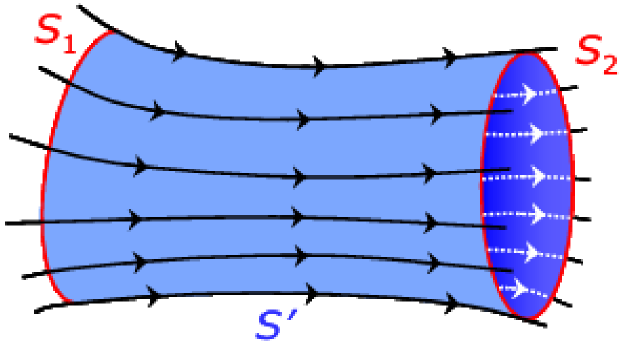
Since the density **ρ** is given by: $\boldsymbol{\rho = \frac{dm}{dV}}$

we obtain the relation between mass flow rate and volume flow rate :

$$\boldsymbol{Q_m = \rho\, Q_V}$$

II.3 Continuity equation

We have seen that all streamlines passing through the same closed contour define a surface called a **streamtube**. If the flow is steady, the mass flow rate is conserved through any cross-section of the streamtube $\boldsymbol{dm_A = dm_B}$



During the infinitely time interval dt, the mass $dm_A = \rho_A S_A dx_A$ of the fluid crosses the section S_A and the mass $dm_B = \rho_B S_B dx_B$ crosses the section S_B.

The conservation of mass gives: $dm_A = dm_B \Rightarrow \rho_A S_A dx_A = \rho_B S_B dx_B$

By dividing by **dt** :

$$\boldsymbol{\rho_A S_A \frac{dx_A}{dt} = \rho_B S_B \frac{dx_B}{dt}}$$ we obtient :

$$\boldsymbol{\rho_A S_A v_A = \rho_B S_B v_B}$$ **with**: $$Q_{m(A)} = Q_{m(B)}$$

$$\boldsymbol{Mass\ flow\ in = Mass\ flow\ out}$$

$$Q_{m(A)} = Q_{m(B)} \qquad\qquad Mass\ flow\ in = Mass\ flow\ out$$

Cette relation traduit ***la conservation du débit massique dans un écoulement permanent***.

- Dans le cas particulier d'un fluide **incompressible** (ρ=cste), on retrouve : $S_A v_A = S_B v_B$

Ce qui est la forme la plus connue de l'équation de continuité.

- This relation expresses **<u>the conservation of mass flow in a steady flow</u>**.
- **The mass flow rate** Q_m is expressed in (Kg/s)

It is defined as the mass of fluid that flows through a given cross-section per unit time.

- The **volume flow rate** Q_v is expressed in (m^3/s)

It is defined as the volume of fluid that flows through a given cross-section per unit time.

Where:

S : area (m^2)

ρ: density of the fluid (Kg/m^3)

v : speed in (m/s)

❖ In the particular case of an incompressible fluid (ρ= cste), we find:

$$S_A v_A = S_B v_B$$

$$\textit{Incoming volume flow} = \textit{Outgoing volume flow}$$

which is the most well-known form of the continuity equation.

This relationship expresses **the conservation of volume flow in a steady flow**.

Note:

For most applications in hydraulics, where water is considered as an incompressible medium, the flow rate is taken in the **volumetric** sense.

In contrast, in aerodynamics (aeraulics), since the gas density varies, only the **mass flow rate** is significant for characterizing the material flux.

nous allons devoir établir les équations générales qui gouvernent l'écoulement.

Tout cela pour quoi ? Afin de prédire la structure de l'écoulement. En fait, on va prédire deux choses, la vitesse et la pression. Nous avons toujours deux grandes inconnues, la vitesse, parce qu'à partir de la vitesse et de la pression, je peux retrouver toutes les autres caractéristiques.

Je peux retrouver les débits, les énergies, etc. Mais pour l'instant, je vais devoir prédire, c'est-à-dire déterminer quelle est la vitesse, la vitesse qui est un vecteur, et quelle est la pression installée. C'est la question à laquelle nous allons essayer d'y répondre durant le chapitre.

- pour pouvoir décrire l'écoulement d'un fluide, on est passé à la description eulerienne. C'est quoi la description eulerienne ? On a un point de l'espace x, y, z, et à un certain temps donné, qu'est-ce qu'il va se passer ? Il va y avoir une vitesse en ce point-là.

c'est-à-dire introduire la notion de l'espace et le temps. Nous allons écrire les théorèmes de conservation de la mécanique classique en variables euleriennes.

Et bien, pour un fluide réel ou bien parfait incompressible, pour lequel il n'y a pas de transfert de matière ni de transfert de chaleur, il y a quatre théorèmes intégraux. Il existe quatre théorèmes intégraux.

To describe the flow of a fluid, we switched to the Eulerian model. What is the Eulerian model?

The **Eulerian description** consists of **studying a fluid flow by observing the physical quantities (velocity, pressure, temperature, density, etc.) at fixed points in space over time**.

In other words, we **do not follow individual fluid particles**; instead, we **observe what happens at a given point in space** as time passes.

We then express the flow quantities as fields depending on position and time:

$$\vec{V} = \vec{V}(x, y, z, t), p = p(x, y, z, t), \rho = \rho(x, y, z, t)$$

That is to say, we introduce the notions of **space** and **time**.

We will now express the **classical mechanics conservation theorems** (mass, momentum, and energy) **in Eulerian variables**.

Comparison: Lagrangian Description

Lagrangian: We follow a **fluid particle** along its motion and observe how its properties change over time.

→ We then write: $\vec{V} = \vec{V}(t)$for a given particle.

Eulerian: We remain **fixed in space** and observe the **fluid passing by**.

→ We write: $\vec{V} = \vec{V}(x, y, z, t)$.

Difference with the Lagrangian Description

Approach	Reference Frame	Observation	Mathematical Expression
Lagrangian	Follow a **fluid particle**	Observe how its properties change over time	$\vec{V} = \vec{V}(t)$for a given particle
Eulerian	Stay at a **fixed point in space**	Observe the fluid passing by	$\vec{V} = \vec{V}(x, y, z, t)$

When adopting the **Eulerian description**, we explicitly introduce the **spatial and temporal dependence** of the fluid's physical quantities (velocity, density, temperature, etc.).

This makes it possible to rewrite the **fundamental principles of classical mechanics** — conservation of mass, momentum, and energy — **in Eulerian variables**, that is, in the form of **local equations valid at every point within the fluid**.

In fluid mechanics, we often want to know **how a fluid particle experiences the variation of its properties over time**, even though our equations are expressed in **Eulerian variables**.

We therefore introduce the **material (or substantial) derivative**:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla$$

This derivative makes it possible to **link the two descriptions**:

The first term, $\partial/\partial t$, represents the **local variation** (at a fixed point in space).

The second term, $\vec{V} \cdot \nabla$,represents the **variation due to the motion of the fluid** (advection).

The notation D/Dt, indicating a Lagrangian derivative (sometimes referred to as a material or substantive derivative), is used in the previous equation to show that we are following a particular element (portion) of fluid and not just a single particle. More generally, we have:

$$\frac{D}{Dt}(\dots) = \frac{\partial(\dots)}{\partial t} + (\mathbf{u} \cdot \nabla)(\dots)$$

•Rate of change (Lagrangian)

•Local variation or Eulerian (effect of time)

•Convective variation due to the change of position of the particle

$$\frac{D}{Dt}(\dots) = \frac{\partial(\dots)}{\partial t} + u_x \frac{\partial(\dots)}{\partial x} + u_y \frac{\partial(\dots)}{\partial y} + u_z \frac{\partial(\dots)}{\partial z}$$

Where (...) represents any quantity of interest in the flow field.

- Conservation of mass (continuity equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0$$

→It states that mass is conserved within every fluid volume element.

b) Conservation of momentum (Navier–Stokes equation)

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{V}$$

where:

• $\rho \frac{D\vec{V}}{Dt}$ = change in momentum,

• $-\nabla p$ = pressure force,

• $\mu \nabla^2 \vec{V}$ = internal viscous effects,

• $\rho \vec{g}$ = body forces (weight).

C- Conservation of energy

$$\rho \frac{De}{Dt} = -p\left(\nabla \cdot \vec{V}\right) + \Phi + \nabla \cdot (k\nabla T)$$

where:

•e = internal energy,

•Φ = viscous dissipation,

•$k\nabla T$ = heat flux by conduction.

Principle	Eulerian equation	Meaning
Mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$	Conservation of mass
Momentum	$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g}$	Second law of Newton applied to the fluid
Energy	$\rho \frac{De}{Dt} = -p(\nabla \cdot \vec{V}) + \Phi + \nabla \cdot (k\nabla T)$	Conservation of energy

The principle of mass conservation is first written for a fixed control volume V surrounded by a closed surface S:

$$\frac{d}{dt}\int_V \rho \ dV = -\oint_S \rho \vec{V} \cdot \vec{n} \, dS$$

Interpretation:

•$\int_V \rho \ dV$:total mass contained in the volume V.

•$\oint_S \rho \vec{V} \cdot \vec{n} \, dS$:mass flow rate exiting the volume.

•The minus sign indicates that the mass decreases if fluid leaves.

This is the integral form of the theorem of mass conservation.

We apply the divergence (Gauss) theorem:

$$\oint_S \rho \vec{V} \cdot \vec{n} \, dS = \int_V \nabla \cdot (\rho \vec{V}) \, dV$$

Thus, conservation becomes:

$$\frac{d}{dt}\int_V \rho \ dV = -\int_V \nabla \cdot (\rho \vec{V}) \, dV$$

Since the volume V is fixed (doesn't depend on time), the derivative can be brought inside the integral sign:

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla \cdot (\rho \vec{V})\, dV$$

Since the equality must be true for any volume V, we can write the local relation (valid at every fluid point):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

This is the continuity equation in Eulerian variables.

In mechanics, the mass of a material system is invariant over time: this is the invariance of mass.

If the density is constant ($\rho = const$), then:

$$\nabla \cdot \vec{V} = 0$$

This relation expresses that the fluid does not compress or expand.

Step	Form	Expression
Global theorem	Integral	$\frac{d}{dt}\int_V \rho\, dV = -\oint_S \rho \vec{V} \cdot \vec{n}\, dS$
After Gauss	Integral over V	$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V})\right) dV = 0$
Local form (Eulerian)	Pointwise	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$

The local Eulerian form (Navier–Stokes equation) is based on the conservation of momentum.

1- Fundamental principle

We start from Newton's principle applied to a fluid:

The change in the momentum of a fluid contained within a control volume equals the sum of the external forces acting on that volume.

2. Integral (global) form

Let V be a fixed control volume, bounded by a closed surface S.

The total momentum of the fluid in V is:

$$\int_V \rho \vec{V}\, dV$$

The conservation theorem is written:

$$\frac{d}{dt}\int_V \rho \vec{V}\, dV = -\oint_S \rho \vec{V}(\vec{V}\cdot\vec{n})\, dS + \oint_S \vec{\tau}\cdot\vec{n}\, dS + \int_V \rho \vec{g}\, dV$$

Explanation of the terms:

- $\frac{d}{dt}\int_V \rho\,\vec{V}\,dV$ → change in the momentum within the volume,
- $-\oint_S \rho\,\vec{V}\left(\vec{V}\cdot\vec{n}\right)dS$ → outgoing momentum flux,
- $\oint_S \vec{\tau}\cdot\vec{n}\,dS$ → surface forces (pressure and viscosity),
- $\int_V \rho\,\vec{g}\,dV$ → body forces (weight, etc.)

4- Details of surface forces

The total stress tensor $\vec{\tau}$ is decomposed as:

$$\vec{\tau} = -p\mathbf{I} + \vec{\tau}_v$$

where:

- p: is the pressure,
- $\mathbf{I}$: identity matrix,
- $\vec{\tau}_v$:viscous stress tensor (internal forces due to viscosity).

Thus:

$$\oint_S \vec{\tau}\cdot\vec{n}\,dS = -\oint_S p\,\vec{n}\,dS + \oint_S \vec{\tau}_v\cdot\vec{n}\,dS$$

4. Application of Gauss's Theorem

We transform the **surface integrals** into **volume integrals**:

Therefore:

$$\frac{d}{dt}\int_V \rho\,\vec{V}\,dV = -\int_V \nabla\cdot\left(\rho\vec{V}\vec{V}\right)dV - \int_V \nabla p\,dV + \int_V \nabla\cdot\vec{\tau}_v\,dV + \int_V \rho\,\vec{g}\,dV$$

Exercise 12: EX01

The velocity field of a two-dimensional flow is defined by:

$$\vec{V} = y\,\vec{\imath} - x\,\vec{\jmath}$$

Determine the equation of the streamline passing through the point $(1, 0)$.

Solution :

L'équation de ligne de courant est donnée par :

$\frac{dy}{dx} = \frac{v}{u}$ D'où, $udy - vdx = 0$

Alors,

$\frac{dy}{dx} = \frac{-x}{y}$ $\rightarrow$ $ydy = -xdx$

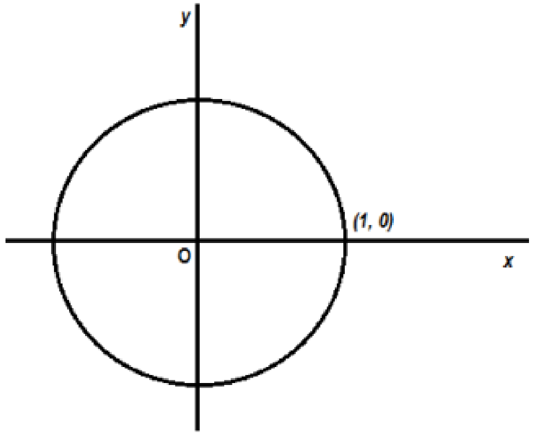
Par intégration, on aura :

$x^2 + y^2 = c$ Où, c : est une constante d'intégration

Pour la ligne de courant passant par le point $(x, y) = (1, 0)$

On trouve que, $c = 1$

Donc, l'équation de ligne de courant est : $x^2 + y^2 = 1$.



Exercise 07: EX 02

The velocity field of a steady, incompressible, planar flow in the $X\text{–}Y$ plane is given by:

$$\vec{V} = a\,x\,\vec{i} - b\,y\,\vec{j} \;with\, a = b = 1\,[s^{-1}]$$

1.Find the equation of the streamlines of the flow.

2.Draw/represent the streamlines for $x \geq 0$and $y \geq 0$.

Solution :

La pente de la ligne de courant dans le plan *X-Y* est donnée par :

$$\frac{dy}{dx} = \frac{v}{u}$$

Pour $\vec{V} = ax\vec{\imath} - by\vec{\jmath}$ On a : $u = ax$, $v = -by$

Alors,

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{by}{ax}$$

Pour résoudre cette équation différentielle, on fait une séparation de variables ensuite on intègre.

$$\int \frac{dy}{y} = -\int \frac{b}{a}\frac{dx}{x}$$

$$lny = -\frac{b}{a} lnx + c$$

$$lny = lnx^{-\frac{b}{a}} + lnc$$ Avec, $c = lnc$

Donc :

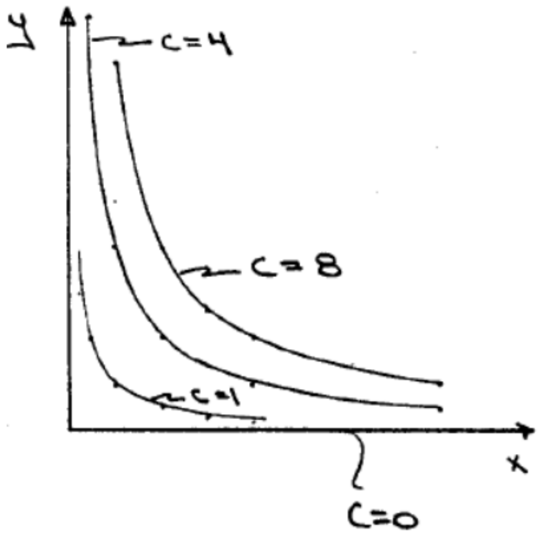
$$y = cx^{-\frac{b}{a}}$$

Pour un champ de vitesse donné, les constantes *a* et *b* sont fixes. Différentes lignes de courant peuvent être obtenues en attribuant différentes valeurs à la constante d'intégration, *c*.

Puisque $a = b = 1\ sec^{-1}$, alors $b / a = 1$ et les lignes de courant sont données par l'équation.

$y = cx^{-1} = \frac{c}{x}$ Il s'agit de lignes hyperboliques.

Pour $c = 0$, $y = 0$ pour tout x et pour $x = 0$ pour tout y.



Exercise 02: EX 03

We consider the flow defined in Eulerian variables by:

$$\begin{cases} u = \omega x \\ v = \omega y \\ w = -\omega x + \alpha t \end{cases}$$

Where ω is not zero.

1.Is this flow stationary (steady), incompressible?

2.Determine the trajectories.

3.Determine the streamlines at instant t_1.

Solution :

1. Si $\alpha \neq 0$, l'écoulement n'est pas stationnaire. Mais si, $\alpha = 0$, l'écoulement est stationnaire.

Un écoulement est incompressible si, $div\vec{V} = 0$.

Alors,

$div\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2w \neq 0$; Donc l'écoulement n'est pas incompressible.

2. Les équations différentielles des trajectoires sont les suivantes :

$$\frac{dx}{\omega x} = \frac{dy}{\omega y} = \frac{dz}{-\omega x + \alpha t} = dt$$

De, $\frac{dx}{\omega x} = dt \quad \rightarrow \frac{dx}{x} = \omega dt$

Par intégration, on obtient : $x = x_0 e^{\omega t}$

De même, $y = y_0 e^{\omega t}$

Si $x = x_0$ *et* $y = y_0$ *en* $t = 0$.

On écrit pour z, $z = -x_0 e^{\omega t} + \alpha \frac{t^2}{2} + k$

Si $z = z_0$ en $t = 0$, alors, $k = x_0 + z_0$. On aura donc :

$$z = x_0(1 - e^{\omega t}) + \alpha \frac{t^2}{2} + z_0$$

Les trois trajectoires recherchées sont exprimés par les relations de x, y et z issues de l'intégration.

Exercise 05: EX04

The velocity field of a flow is given by:

$$\vec{V} = Axy\,\vec{\imath} + By^2\,\vec{\jmath}$$

Given: $A = 1\,[1/m.s]$ and $B = -0.5\,[1/m.s]$

1.Check if this flow is conservative.

2.Determine the equation of the streamlines of this flow.

3.Plot the stream function for: $x, y \geq 0$

Solution :

1. Nous devons montrer que, $div\vec{V} = 0$.

Pour cela, il nous suffit de vérifier que l'équation suivante est vraie :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

D'après le champ de vitesse donné, on a :

$$\frac{\partial Axy}{\partial x} + \frac{\partial By^2}{\partial y} = Ay + 2By = y - y = 0$$

L'équation de continuité est vérifiée, le fluide est bien conservatif.

2. Les lignes de courant sont tangentes aux vecteurs de vitesses, on écrit donc :

$$\frac{dy}{dx} = \frac{v}{u}$$

Pour $\vec{V} = Axy\vec{\imath} + By^2\vec{\jmath}$ On a : $u = Axy,\quad v = By^2$

Alors,

$$\frac{dy}{dx} = \frac{v}{u} = \frac{By^2}{Axy} = \frac{By}{Ax}$$

Pour résoudre cette équation différentielle, on fait une séparation de variables ensuite on intègre.

$$\int \frac{dy}{y} = \int \frac{B}{A}\frac{dx}{x}$$

$$lny = \frac{B}{A}lnx + c$$

$$lny = lnx^{\frac{B}{A}} + lnc \qquad \text{Avec, } c = lnc$$

Donc :

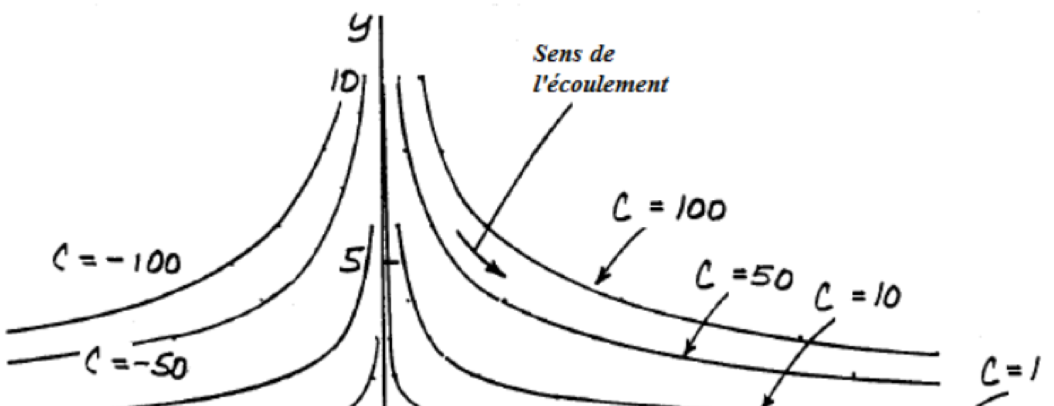
$$y = cx^{\frac{B}{A}}$$

3. Pour un champ de vitesse donné, les constantes *A* et *B* sont fixes. Différentes lignes de courant peuvent être obtenues en attribuant différentes valeurs à la constante d'intégration, *c*.

Puisque $A = 1\ 1/m.s$, et $B = -0.5\ 1/m.s$ alors $\frac{B}{A} = -\frac{0.5}{1.0} = -\frac{1}{2}$ et les lignes de courant sont données par l'équation.

$$y = cx^{-\frac{1}{2}} = \frac{c}{x^{\frac{1}{2}}}$$ Il s'agit de lignes hyperboliques.

Pour $c = 0$, $y = 0$ pour tout x et pour $x = 0$ pour tout y.



Exercise 11:

For the following steady, incompressible, and planar flows in XY:

a. $\vec{V}$

$$\begin{cases} u = x^2 + y^2 \\ v = -2xy \end{cases}$$

b. $\vec{V}$

$$\begin{cases} u = 9xy + y \\ v = 8xy + 2x \end{cases}$$

Verify whether there is mass conservation for these flows.

Solution :

La conservation de masse est satisfaite si :

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$ (Eq. de continuité)

a. $\frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(-2xy) = 0 \Rightarrow \quad 2x\text{-}2x=0$

La masse est donc conservée. Un tel écoulement est possible. Le champ de vitesse est représenté ci-dessous.

b. $\frac{\partial}{\partial x}(9xy + y) + \frac{\partial}{\partial y}(8xy + 2x) = 9y + 8x \neq 0,$

Exercise 08: EX05

The velocity field of a flow is given by:

$$\vec{V} = ax^2\vec{\imath} + bxy\vec{\jmath} \text{ with } a = 2\,[1/m.s];\; b = -4\,[1/m.s].$$

1. Is this flow steady (permanent)?
2. What is the type of the flow? Why?
3. Determine the velocity components at the point $(x, y, z) = \left(2, \frac{1}{2}, 0\right)$.
4. Determine the equation of streamlines $y = f(x)$ of the flow passing through the previous point.

Solution :

1. D'après le champ de vitesse donné, $\frac{d\vec{V}}{dt} = 0$, donc l'écoulement est permanent.

2. D'après le champ de vitesse donnée : $\vec{V} = ax^2\vec{\imath} + bxy\vec{\jmath}$ On peut dire que l'écoulement est bidimensionnel (plan), parce qu'on voit que le vecteur vitesse n'a que deux composantes,

$$u = ax^2 \text{ et } v = bxy,\, w = 0$$

3. La vitesse au point (x, y, z) = (2, ½, 0)

$$\vec{V} = ax^2\vec{\imath} + bxy\vec{\jmath} = 2(2^2)\vec{\imath} - 4(2)\left(\frac{1}{2}\right)\vec{\jmath} = 8\vec{\imath} - 4\vec{\jmath}$$

D'où : $\begin{cases} u = 8\,m/s \\ v = -4\,m/s \end{cases}$

4. La pente de la ligne de courant dans le plan *X-Y* est donnée par :

$$\frac{dy}{dx} = \frac{v}{u}$$

Pour $\vec{V} = ax^{2\vec{\imath}} + bxy\vec{\jmath}$ On a : $u = ax^2$, $v = bxy$

Alors,

$$\frac{dy}{dx} = \frac{v}{u} = \frac{bxy}{ax^2} = \frac{by}{ax}$$

Pour résoudre cette équation différentielle, on fait une séparation de variables ensuite on intègre.

$$\int \frac{dy}{y} = \int \frac{b}{a}\frac{dx}{x}$$

$$lny = \frac{b}{a} lnx + c$$

$lny = lnx^{\frac{b}{a}} + lnc$ Avec, $c = lnc$

Donc :

$y = cx^{\frac{b}{a}}$ Avec, $\frac{b}{a} = -2$, il vient donc : $y = \frac{c}{x^2}$

Au point, (x, y, z) = (2, ½, 0), $c = x^2y = (2)^2\left(\frac{1}{2}\right) = 2\,m^3$

Alors, $y = \frac{2}{x^2}$

5. Plusieurs lignes de courant peuvent être obtenues en attribuant différentes valeurs à la constante d'intégration, *c*.

Pour résoudre cette équation différentielle, on fait une séparation de variables ensuite on intègre.

$$\int \frac{dy}{y} = \int \frac{b}{a}\frac{dx}{x}$$

$$lny = \frac{b}{a} lnx + c$$

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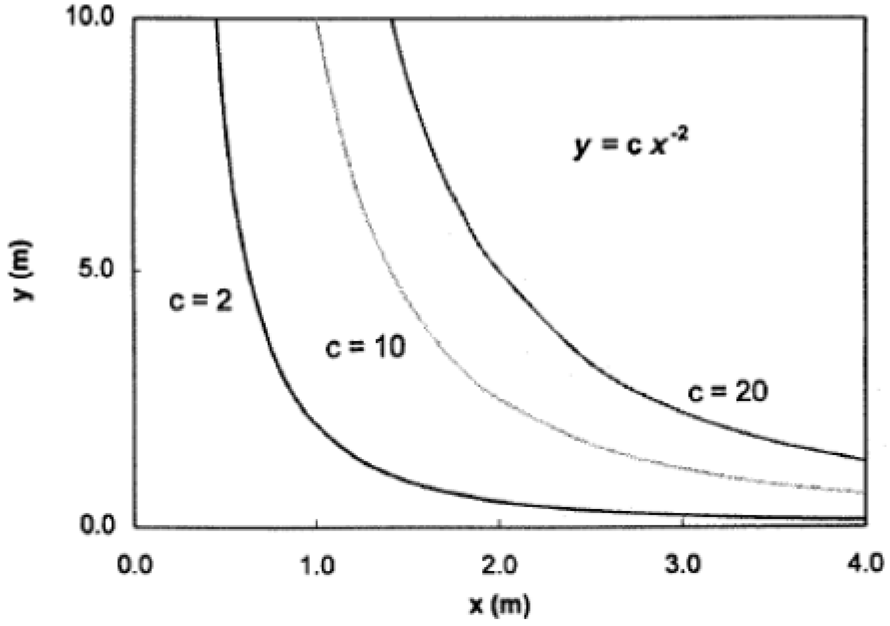

Exercise 01:

We consider the flow defined in Lagrangian variables by:

$$\begin{cases} x = a + \alpha t \\ y = b + \beta t^2 \\ z = c + \gamma t^3 + c\alpha t \end{cases}$$

Give the velocity $\vec{V}$ of this flow in Eulerian variables.

Solution :

On vérifie qu'en $t = 0$, on a : $\begin{cases} x = a \\ y = b \\ z = c \end{cases}$

Les variables (a, b, c, t) sont les variables de Lagrange. Exprimons la vitesse en variables de Lagrange :

$u = \frac{dx}{dt} = \alpha$; $v = \frac{dy}{dt} = 2\beta t$; $w = \frac{dz}{dt} = 3\gamma t^2 + c\alpha$

La vitesse $\vec{V}$ (u, v, w) est une variable d'Euler. Elle est exprimée en fonction de a, b et c.

Essayons de l'exprimer en fonction des variables (x, y, z) de Lagrange.

Sur la base des données de l'énoncé, on a :

$$a = x - \alpha t\ ;\quad b = y - \beta t^2\ ;\qquad c = \frac{z - \gamma t^3}{1 + \alpha t}$$

En portant ces expressions dans ceux de (u, v, w), on obtient :

$$\begin{cases} u = \alpha \\ v = 2\beta t \\ w = 3\gamma t^2 + \dfrac{\alpha(z - \gamma t^3)}{1 + \alpha t} \end{cases}$$