

# Analysis I: Solutions of Tutorial Exercise Sheet 4

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## Exercise 1: Comparison

For each function  $f(x)$  as  $x \rightarrow 0$ , we examine the order relative to  $x$  by studying the limit

$$L = \lim_{x \rightarrow 0} \frac{f(x)}{x},$$

when it exists, or by bounding the ratio. According to the definitions:

- If  $L$  is a finite nonzero number, then  $f(x)$  is of the **same order** as  $x$ .
- If  $L = 0$ , then  $f(x)$  is of **higher order** than  $x$ .
- If  $L = \infty$  (or  $|f(x)/x| \rightarrow \infty$ ), then  $f(x)$  is of **lower order** than  $x$ .

We now treat each case.

1.  $f(x) = x^2$ .

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

Hence,  $x^2$  is of higher order than  $x$ . In Landau notation:

$$x^2 = o(x) \quad (x \rightarrow 0).$$

2.  $f(x) = \sqrt{x^2 + x^4}$ . Write  $\sqrt{x^2 + x^4} = |x|\sqrt{1 + x^2}$ . Then

$$\frac{f(x)}{x} = \frac{|x|}{x} \sqrt{1 + x^2}.$$

The two-sided limit does not exist because  $|x|/x$  is  $+1$  for  $x > 0$  and  $-1$  for  $x < 0$ . However,

$$\lim_{x \rightarrow 0} \left| \frac{f(x)}{x} \right| = \lim_{x \rightarrow 0} \sqrt{1 + x^2} = 1,$$

and for sufficiently small  $x$  we have

$$|x| \leq |f(x)| \leq \sqrt{2}|x|.$$

Therefore,  $f(x)$  is of the same order as  $x$ . In Landau notation:

$$\sqrt{x^2 + x^4} = O(x) \quad (x \rightarrow 0).$$

3.  $f(x) = \sin(3x)$ .

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3 \neq 0.$$

Thus,  $\sin(3x)$  is of the same order as  $x$ . In Landau notation:

$$\sin(3x) = O(x) \quad (x \rightarrow 0).$$

4.  $f(x) = 2x \cos(x^3) \tan^2(x)$ . As  $x \rightarrow 0$ , we have  $\cos(x^3) \rightarrow 1$  and  $\tan(x) \sim x$ , so  $\tan^2(x) \sim x^2$ . Consequently,

$$f(x) \sim 2x \cdot 1 \cdot x^2 = 2x^3.$$

Then

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} 2 \cos(x^3) \tan^2(x) = 0.$$

Hence,  $f(x)$  is of higher order than  $x$ . In Landau notation:

$$2x \cos(x^3) \tan^2(x) = o(x) \quad (x \rightarrow 0).$$

5.  $f(x) = e^{2x}$ . As  $x \rightarrow 0$ ,  $e^{2x} \rightarrow 1$  while  $x \rightarrow 0$ . Therefore,

$$\lim_{x \rightarrow 0} \frac{e^{2x}}{x} = \infty.$$

So  $e^{2x}$  is of lower order than  $x$ . Since it tends to a nonzero constant, it is of order zero. In Landau notation:

$$x = o(e^{2x}) \quad (x \rightarrow 0).$$

## Exercise 2: Polynomial Expansion

1. Expand the polynomial  $P(x) = x^4 - 5x^3 + 5x^2 + x + 2$  in powers of  $x - 2$ .

We compute the Taylor expansion around  $a = 2$ :

$$P(x) = \sum_{k=0}^4 \frac{P^{(k)}(2)}{k!} (x-2)^k.$$

The derivatives are:

$$\begin{aligned}P(x) &= x^4 - 5x^3 + 5x^2 + x + 2, \\P'(x) &= 4x^3 - 15x^2 + 10x + 1, \\P''(x) &= 12x^2 - 30x + 10, \\P'''(x) &= 24x - 30, \\P^{(4)}(x) &= 24.\end{aligned}$$

Evaluating at  $x = 2$ :

$$\begin{aligned}P(2) &= 0, \\P'(2) &= -7, \\P''(2) &= -2, \\P'''(2) &= 18, \\P^{(4)}(2) &= 24.\end{aligned}$$

Hence,

$$\begin{aligned}P(x) &= \frac{0}{0!} + \frac{-7}{1!}(x-2) + \frac{-2}{2!}(x-2)^2 + \frac{18}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4 \\&= -7(x-2) - (x-2)^2 + 3(x-2)^3 + (x-2)^4.\end{aligned}$$

Thus,

$$\boxed{x^4 - 5x^3 + 5x^2 + x + 2 = (x-2)^4 + 3(x-2)^3 - (x-2)^2 - 7(x-2)}.$$

2. Expand the polynomial  $Q(x) = x^5 + 2x^4 - x^2 + x + 1$  in powers of  $x + 1$ .

Expanding around  $a = -1$ :

$$Q(x) = \sum_{k=0}^5 \frac{Q^{(k)}(-1)}{k!} (x+1)^k.$$

The derivatives are:

$$\begin{aligned}Q(x) &= x^5 + 2x^4 - x^2 + x + 1, \\Q'(x) &= 5x^4 + 8x^3 - 2x + 1, \\Q''(x) &= 20x^3 + 24x^2 - 2, \\Q'''(x) &= 60x^2 + 48x, \\Q^{(4)}(x) &= 120x + 48, \\Q^{(5)}(x) &= 120.\end{aligned}$$

Evaluating at  $x = -1$ :

$$\begin{aligned} Q(-1) &= 0, \\ Q'(-1) &= 0, \\ Q''(-1) &= 2, \\ Q'''(-1) &= 12, \\ Q^{(4)}(-1) &= -72, \\ Q^{(5)}(-1) &= 120. \end{aligned}$$

Hence,

$$\begin{aligned} Q(x) &= \frac{0}{0!} + \frac{0}{1!}(x+1) + \frac{2}{2!}(x+1)^2 + \frac{12}{3!}(x+1)^3 + \frac{-72}{4!}(x+1)^4 + \frac{120}{5!}(x+1)^5 \\ &= (x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5. \end{aligned}$$

Thus,

$$\boxed{x^5 + 2x^4 - x^2 + x + 1 = (x+1)^5 - 3(x+1)^4 + 2(x+1)^3 + (x+1)^2}.$$

### Exercise 3: Taylor's Formula for $f(x) = \sqrt{x}$ at $a = 1$ , $n = 3$

Taylor's formula with Lagrange remainder for a function  $f$  that is  $n + 1$  times differentiable is:

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1},$$

and  $\xi$  lies between  $a$  and  $x$ .

For  $f(x) = \sqrt{x} = x^{1/2}$ ,  $a = 1$ , and  $n = 3$ , we compute the derivatives:

$$\begin{aligned} f(x) &= x^{1/2}, & f(1) &= 1, \\ f'(x) &= \frac{1}{2}x^{-1/2}, & f'(1) &= \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}x^{-3/2}, & f''(1) &= -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}x^{-5/2}, & f'''(1) &= \frac{3}{8}, \\ f^{(4)}(x) &= -\frac{15}{16}x^{-7/2}. \end{aligned}$$

The Taylor polynomial of degree 3 is:

$$P_3(x) = 1 + \frac{1}{2}(x-1) + \frac{-\frac{1}{4}}{2!}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

The remainder term in Lagrange form is:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!} (x-1)^4 = \frac{-\frac{15}{16}\xi^{-7/2}}{24} (x-1)^4 = -\frac{5}{128} \frac{(x-1)^4}{\xi^{7/2}},$$

where  $\xi$  is a number between 1 and  $x$ . Equivalently, we can write  $\xi = 1 + \theta(x-1)$  with  $0 < \theta < 1$ .

Thus, the complete Taylor formula for  $\sqrt{x}$  is:

$$\boxed{\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128} \frac{(x-1)^4}{(1+\theta(x-1))^{7/2}}, \quad 0 < \theta < 1.}$$

### Exercise 4: Maclaurin Formula for $f(x) = \sqrt{1+x}$

- (1) Give the Maclaurin formula for  $f(x) = \sqrt{1+x}$  when  $n = 2$ .

The Maclaurin formula (Taylor expansion at  $a = 0$ ) with Lagrange remainder is

$$f(x) = \sum_{k=0}^2 \frac{f^{(k)}(0)}{k!} x^k + R_2(x),$$

where

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!} x^3,$$

and  $\xi$  lies between 0 and  $x$  (equivalently  $\xi = \theta x$  with  $0 < \theta < 1$ ).

Compute the derivatives:

$$\begin{aligned} f(x) &= (1+x)^{1/2}, & f(0) &= 1, \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f'(0) &= \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2}, & f''(0) &= -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2}. \end{aligned}$$

Hence,

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{-\frac{1}{4}}{2!}x^2 + R_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + R_2(x),$$

with

$$R_2(x) = \frac{1}{3!} \frac{3}{8} (1+\xi)^{-5/2} x^3 = \frac{1}{16} (1+\xi)^{-5/2} x^3, \quad 0 < \xi < x.$$

Writing  $\xi = \theta x$  ( $0 < \theta < 1$ ), we obtain the formula

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}(1+\theta x)^{-5/2}x^3, \quad 0 < \theta < 1.}$$

- (2) Give the error of the approximation  $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$  when  $x = 0.2$ .

The error is exactly the remainder  $R_2(0.2)$ . For  $x = 0.2$ ,

$$R_2(0.2) = \frac{1}{16} (1+0.2\theta)^{-5/2} (0.2)^3 = \frac{0.008}{16} (1+0.2\theta)^{-5/2} = 0.0005 (1+0.2\theta)^{-5/2},$$

with  $0 < \theta < 1$ .

The factor  $(1+0.2\theta)^{-5/2}$  is a decreasing function of  $\theta$ . Therefore,

$$(1+0.2)^{-5/2} < (1+0.2\theta)^{-5/2} < 1,$$

i.e.,

$$0.634 \approx (1.2)^{-5/2} < (1+0.2\theta)^{-5/2} < 1.$$

Consequently,

$$0.0005 \times 0.634 < R_2(0.2) < 0.0005,$$

or

$$3.17 \times 10^{-4} < R_2(0.2) < 5.00 \times 10^{-4}.$$

Using the exact value  $\sqrt{1.2} \approx 1.095445$  and the approximation  $1 + 0.1 - 0.005 = 1.095$ , the actual error is approximately

$$1.095445 - 1.095 = 0.000445.$$

Thus, for  $x = 0.2$ , the error of the given approximation is about

$$\boxed{4.45 \times 10^{-4}}.$$

(Equivalently, one can say that the error lies between  $3.17 \times 10^{-4}$  and  $5.00 \times 10^{-4}$ .)

## Exercise 5: Taylor Expansion of Sine

(1) Show that

$$\sin x = \sin a + (\cos a)(x - a) - \frac{(\sin a)(x - a)^2}{2!} - \frac{(\cos \xi)(x - a)^3}{3!}, \quad a < \xi < x.$$

**Solution.** The Taylor expansion of a function  $f$  about the point  $a$  with Lagrange remainder is

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(\xi)}{3!}(x - a)^3,$$

where  $\xi$  lies between  $a$  and  $x$ . For  $f(x) = \sin x$  we have

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x.$$

Substituting,

$$\sin x = \sin a + \cos a(x - a) + \frac{-\sin a}{2!}(x - a)^2 + \frac{-\cos \xi}{3!}(x - a)^3,$$

which is exactly the required formula.

(2) Use the formula to approximate  $\sin 49^\circ$  and estimate the error.

**Solution.** Choose  $a = 45^\circ$  (a point where sine and cosine are known exactly) and  $x = 49^\circ$ . Convert to radians:

$$a = 45^\circ = \frac{\pi}{4}, \quad x = 49^\circ = \frac{49\pi}{180}, \quad x - a = 4^\circ = \frac{\pi}{45}.$$

Also,

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

Using the expansion up to the quadratic term (neglecting the remainder), we obtain

$$\begin{aligned} \sin 49^\circ &\approx \sin 45^\circ + \cos 45^\circ \cdot \frac{\pi}{45} - \frac{\sin 45^\circ}{2!} \left(\frac{\pi}{45}\right)^2 \\ &= \frac{\sqrt{2}}{2} \left(1 + \frac{\pi}{45} - \frac{1}{2} \left(\frac{\pi}{45}\right)^2\right). \end{aligned}$$

Numerical evaluation:

$$\frac{\pi}{45} \approx 0.06981317, \quad \left(\frac{\pi}{45}\right)^2 \approx 0.004873, \quad \frac{1}{2} \left(\frac{\pi}{45}\right)^2 \approx 0.0024365.$$

Hence,

$$1 + \frac{\pi}{45} - \frac{1}{2} \left(\frac{\pi}{45}\right)^2 \approx 1.06737667,$$

and

$$\sin 49^\circ \approx 0.70710678 \times 1.06737667 \approx 0.7547.$$

(The exact value to six decimals is  $\sin 49^\circ \approx 0.754710$ .)

The error of this approximation is given by the Lagrange remainder:

$$R = -\frac{\cos \xi}{6} \left(\frac{\pi}{45}\right)^3, \quad 45^\circ < \xi < 49^\circ.$$

Since  $\cos$  is positive and decreasing on  $[45^\circ, 49^\circ]$ , the maximum of  $|\cos \xi|$  occurs at  $\xi = 45^\circ$ :

$$|\cos \xi| \leq \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

Therefore,

$$|R| \leq \frac{\sqrt{2}/2}{6} \left(\frac{\pi}{45}\right)^3 = \frac{\sqrt{2}}{12} \left(\frac{\pi}{45}\right)^3.$$

Compute:

$$\left(\frac{\pi}{45}\right)^3 \approx (0.06981317)^3 \approx 0.0003401, \quad \frac{\sqrt{2}}{12} \approx 0.11785,$$

so

$$|R| \lesssim 0.11785 \times 0.0003401 \approx 4.01 \times 10^{-5} < 5 \times 10^{-5}.$$

Hence the absolute error is smaller than 0.00005.

## Exercise 6: Maclaurin Series of $e^x \sin x$

The Maclaurin series expansion of a function  $f(x)$  is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

For  $f(x) = e^x \sin x$ , one can show (by induction) that

$$f^{(n)}(x) = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right).$$

Hence,

$$f^{(n)}(0) = (\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right).$$

Therefore, the Maclaurin series is

$$e^x \sin x = \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right)}{n!} x^n.$$

Explicitly computing the first few coefficients:

$$n = 0 : \sin 0 = 0,$$

$$n = 1 : \sqrt{2} \sin \frac{\pi}{4} = 1,$$

$$n = 2 : 2 \sin \frac{\pi}{2} = 2,$$

$$n = 3 : 2\sqrt{2} \sin \frac{3\pi}{4} = 2,$$

$$n = 4 : 4 \sin \pi = 0,$$

$$n = 5 : 4\sqrt{2} \sin \frac{5\pi}{4} = -4,$$

$$n = 6 : 8 \sin \frac{3\pi}{2} = -8,$$

$$n = 7 : 8\sqrt{2} \sin \frac{7\pi}{4} = -8, \text{ etc.}$$

Dividing by the corresponding factorials gives the series

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \frac{1}{630}x^7 + \dots$$

(One may also obtain this series by multiplying the Maclaurin series of  $e^x$  and  $\sin x$ .)

## Exercise 7: Inequalities via Taylor Expansions

### (a) Maclaurin expansion of order $n$ for $e^x$

The Maclaurin expansion of  $f(x) = e^x$  is given by:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}, \quad \text{with } 0 < \theta < 1.$$

For  $x \geq 0$ , the remainder term is non-negative:

$$\frac{x^{n+1}}{(n+1)!}e^{\theta x} \geq 0.$$

Therefore,

$$e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}.$$

### (b) Approximation for $n = 2$ at $x = 1$

For  $n = 2$ , the expansion is:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{6}e^{\theta x}, \quad \text{with } 0 < \theta < 1.$$

Setting  $x = 1$ , we obtain:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6}e^{\theta} = \frac{5}{2} + \frac{1}{6}e^{\theta}.$$

Since  $0 < \theta < 1$ , we know that  $1 < e^{\theta} < e$ . We apply this to the equation:

$$\frac{5}{2} + \frac{1}{6}(1) < \frac{5}{2} + \frac{1}{6}e^{\theta} < \frac{5}{2} + \frac{1}{6}e.$$

Substituting  $e$  back into the middle term:

$$\frac{5}{2} + \frac{1}{6} < e < \frac{5}{2} + \frac{1}{6}e.$$

From the left side:

$$e > \frac{15}{6} + \frac{1}{6} = \frac{16}{6} = \frac{8}{3}.$$

From the right side:

$$e < \frac{5}{2} + \frac{1}{6}e \implies \frac{5}{6}e < \frac{5}{2} \implies e < 3.$$

Thus, we conclude:

$$\boxed{\frac{8}{3} < e < 3}.$$

### (c) General estimation for $x = 1$

Using the expansion for  $x = 1$  at order  $n$ :

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e^{\theta}}{(n+1)!}, \quad \text{with } 0 < \theta < 1.$$

Rearranging for the remainder:

$$e - \left[ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right] = \frac{e^\theta}{(n+1)!}.$$

Using the bounds  $1 < e^\theta < e < 3$  established previously:

$$\frac{1}{(n+1)!} < \frac{e^\theta}{(n+1)!} < \frac{3}{(n+1)!}.$$

Therefore:

$$\frac{1}{(n+1)!} < e - \sum_{k=0}^n \frac{1}{k!} < \frac{3}{(n+1)!}.$$

## Question 2: Further Maclaurin Applications

### 1) Maclaurin formula of order 4 for $\cos(x)$

$$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} \sin(\theta x), \quad \text{with } 0 < \theta < 1.$$

Consider  $x \in [0, \frac{\pi}{2}]$ . Then  $\theta x \in [0, \frac{\pi}{2}]$ , which implies  $\sin(\theta x) \geq 0$ . Consequently, the remainder term is non-positive:

$$-\frac{x^5}{5!} \sin(\theta x) \leq 0.$$

Thus:

$$\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

### 2) Inequalities for $\ln(x+1)$

We examine the expansions for orders  $n = 3$  and  $n = 4$  for  $x > 0$ .

**Order  $n = 3$ :**

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(\theta x + 1)^4}.$$

Since  $x > 0$ , the remainder term is negative:

$$-\frac{x^4}{4(\theta x + 1)^4} < 0 \implies \ln(x+1) < x - \frac{x^2}{2} + \frac{x^3}{3}.$$

**Order  $n = 4$ :**

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5(\theta x + 1)^5}.$$

Since  $x > 0$ , the remainder term is positive:

$$\frac{x^5}{5(\theta x + 1)^5} > 0 \implies \ln(x+1) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

Combining these results, for  $x > 0$ :

$$\boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} < \ln(x+1) < x - \frac{x^2}{2} + \frac{x^3}{3}}.$$

## Exercise 8: Special Relativity Kinetic Energy

### 1. Classical Limit:

The relativistic kinetic energy is given by:

$$K = m_0 c^2 \left[ \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right].$$

Using the binomial expansion  $(1 - x)^{-1/2} = 1 + \frac{1}{2}x + \mathcal{O}(x^2)$  with  $x = v^2/c^2$ :

$$K = m_0 c^2 \left[ \left( 1 + \frac{1}{2} \frac{v^2}{c^2} \right) - 1 \right] = m_0 c^2 \left( \frac{v^2}{2c^2} \right).$$

Simplifying yields the classical Newtonian kinetic energy:

$$\boxed{K \approx \frac{1}{2} m_0 v^2}.$$

### 2. Error Estimation (Taylor's Formula):

Let  $f(x) = (1 - x)^{-1/2}$  where  $x = v^2/c^2$ . The Taylor expansion of order 1 with the Lagrange remainder is:

$$f(x) = 1 + \frac{1}{2}x + \frac{f''(\xi)}{2!}x^2, \quad \text{where } 0 < \xi < x.$$

Calculating the second derivative,  $f''(x) = \frac{3}{4}(1 - x)^{-5/2}$ . The difference between the relativistic and classical energy ( $\Delta K$ ) corresponds to the remainder term:

$$\Delta K = m_0 c^2 \left( \frac{f''(\xi)}{2!} x^2 \right) = m_0 c^2 \left( \frac{3}{8} (1 - \xi)^{-5/2} \left( \frac{v^2}{c^2} \right)^2 \right).$$

$$\Delta K = \frac{3}{8} m_0 \frac{v^4}{c^2} (1 - \xi)^{-5/2}.$$

**Numerical Evaluation:** Given  $|v| \leq 100$  m/s and  $c \approx 3 \times 10^8$  m/s, we have  $x = (v/c)^2 \approx 1.1 \times 10^{-13}$ . Since  $0 < \xi < x \ll 1$ , we approximate  $(1 - \xi)^{-5/2} \approx 1$ .

$$\Delta K \approx \frac{3}{8} m_0 \frac{(100)^4}{(3 \times 10^8)^2} = \frac{3}{8} m_0 \frac{10^8}{9 \times 10^{16}}.$$

$$\Delta K \approx 4.17 \times 10^{-10} m_0 \text{ J}.$$

This discrepancy is negligible for macroscopic objects at non-relativistic speeds.