Exercise 1

Write the negation of the following propositions:

1. All students in the mathematics department are intelligent.
2. There exists at least one foreign student at Mila University.
3. For every $\varepsilon > 0$, there exists $\eta \in \mathbb{Q}$ such that $0 < \eta < \varepsilon$.
4. For every $x \in \mathbb{R}$, we have $x^2 > 0$.

Solution

1. **Negation:** There exists at least one student in the mathematics department who is not intelligent.
2. **Negation:** All students at Mila University are not foreign. (Or: No student at Batna 2 University is foreign.
3. **Negation:** There exists $x \in \mathbb{R}$ such that $x^2 \leq 0$. (In practice, this is equivalent to $x = 0$ since $x^2 = 0$.)

Exercise 2

Write the following propositions using quantifiers:

1. f is the zero function (where f is a function from $\mathbb{R}$ to $\mathbb{R}$).
2. The denominator D of f vanishes at least once on $\mathbb{R}$.
3. f is the identity function on $\mathbb{R}$ (i.e., the function that maps each real number to itself).
4. The graph of f intersects the line with equation $y = x$.
5. f is increasing on $\mathbb{R}$ (where f is a function from $\mathbb{R}$ to $\mathbb{R}$).

6. The equation $\sin x = x$ has exactly one solution in $\mathbb{R}$.
7. For every point M in the plane $\mathcal{P}$, M lies on the circle $\mathcal{C}$ with center Ω and radius R if and only if the distance from M to Ω equals R.

Solution

1. $\forall x \in \mathbb{R},\ f(x) = 0$
2. $\exists x \in \mathbb{R},\ D(x) = 0$
3. $\forall x \in \mathbb{R},\ f(x) = x$
4. $\exists x \in \mathbb{R},\ f(x) = x$
5. $\forall x, y \in \mathbb{R},\ (x \leq y \ \Rightarrow\ f(x) \leq f(y))$
6. $\exists!\, x \in \mathbb{R},\ \sin x = x$
   Alternatively, in more detail:
   $(\exists x \in \mathbb{R},\ \sin x = x)\ \wedge\ (\forall x, y \in \mathbb{R},\ (\sin x = x\ \wedge\ \sin y = y)\ \Rightarrow\ x = y)$
7. $\forall M \in \mathcal{P},\ (M \in \mathcal{C}\ \Leftrightarrow\ d(M, \Omega) = R)$

Exercise 3

Translate the following assertions into Predicate Logic:

1. All professors are intelligent.
2. There exists an intelligent professor.
3. If a professor teaches Programming Languages (PL), then they are intelligent.
4. Everyone loves everyone.
5. Everyone loves ice cream.
6. Everyone loves someone.
7. Someone loves everyone.
8. Some students attend all classes.

Solution

We use the following predicates:

- $P(x)$: x is a professor
- $I(x)$: x is intelligent
- $T(x)$: x teaches Programming Languages (PL)
- $L(x, y)$: x loves y
- $S(x)$: x is a student
- $A(x, y)$: x attends y (where y is a class)
- $C(x)$: x is ice cream
- The domain for x, y is people unless otherwise specified

1. $\forall x(P(x) \to I(x))$
2. $\exists x(P(x) \land I(x))$
3. $\forall x((P(x) \land T(x)) \to I(x))$
4. $\forall x \forall y L(x, y)$
5. $\forall x \exists y(C(y) \land L(x, y))$
   (Assuming $C(y)$ means y is ice cream and y is in a separate domain of things)
6. $\forall x \exists y L(x, y)$
7. $\exists x \forall y L(x, y)$
8. $\exists x(S(x) \land \forall y A(x, y))$
   (Assuming y ranges over all classes)

Alternative notations: For statements 5-8, some alternative formulations are possible:

- For statement 5: If we want to specify that ice cream is a specific object, we could use: $\forall x L(x, \text{ice\_cream})$
- For statement 8: More precisely: $\exists x(S(x) \land \forall y(\text{Class}(y) \to A(x, y)))$

Exercise 4

Put the following formulas into prenex form:

a) $(\forall x\, P(x) \wedge \exists x\, Q(x)) \rightarrow \forall x\, (R(x) \rightarrow S(x))$

b) $\neg((\forall x\, P(x) \wedge \exists x\, Q(x)) \rightarrow \forall x\, (R(x) \vee \neg S(x)))$

c) $\forall x\, P(x) \leftrightarrow \exists x\, Q(x)$

Solution

a)

$$\begin{aligned} & (\forall x\, P(x) \wedge \exists x\, Q(x)) \rightarrow \forall x\, (R(x) \rightarrow S(x)) \\ \equiv\ & \neg(\forall x\, P(x) \wedge \exists x\, Q(x)) \vee \forall x\, (R(x) \rightarrow S(x)) \\ \equiv\ & (\neg\forall x\, P(x) \vee \neg\exists x\, Q(x)) \vee \forall x\, (\neg R(x) \vee S(x)) \\ \equiv\ & (\exists x\, \neg P(x) \vee \forall x\, \neg Q(x)) \vee \forall x\, (\neg R(x) \vee S(x)) \\ \equiv\ & \exists x\, \neg P(x) \vee \forall x\, \neg Q(x) \vee \forall x\, (\neg R(x) \vee S(x)) \end{aligned}$$

Rename variables: Let $\forall x\, \neg Q(x)$ become $\forall y\, \neg Q(y)$ and $\forall x\, (\neg R(x) \vee S(x))$ become $\forall z\, (\neg R(z) \vee S(z))$. Then:

$$\exists x\, \neg P(x) \vee \forall y\, \neg Q(y) \vee \forall z\, (\neg R(z) \vee S(z))$$

Pull quantifiers to the front (order: existential then universals):

$$\exists x\, \forall y\, \forall z\, (\neg P(x) \vee \neg Q(y) \vee \neg R(z) \vee S(z))$$

So the prenex form is: $\exists x\, \forall y\, \forall z\, (\neg P(x) \vee \neg Q(y) \vee \neg R(z) \vee S(z))$.

b)

$$\begin{aligned} & \neg((\forall x\, P(x) \wedge \exists x\, Q(x)) \rightarrow \forall x\, (R(x) \vee \neg S(x))) \\ \equiv\ & \neg(\neg(\forall x\, P(x) \wedge \exists x\, Q(x)) \vee \forall x\, (R(x) \vee \neg S(x))) \\ \equiv\ & (\forall x\, P(x) \wedge \exists x\, Q(x)) \wedge \neg\forall x\, (R(x) \vee \neg S(x)) \\ \equiv\ & (\forall x\, P(x) \wedge \exists x\, Q(x)) \wedge \exists x\, \neg(R(x) \vee \neg S(x)) \\ \equiv\ & (\forall x\, P(x) \wedge \exists x\, Q(x)) \wedge \exists x\, (\neg R(x) \wedge S(x)) \end{aligned}$$

Rename variables: Let $\forall x\, P(x)$ become $\forall u\, P(u)$, $\exists x\, Q(x)$ become $\exists v\, Q(v)$, and $\exists x\, (\neg R(x) \land S(x))$ become $\exists w\, (\neg R(w) \land S(w))$. Then:

$$\forall u\, P(u) \land \exists v\, Q(v) \land \exists w\, (\neg R(w) \land S(w))$$

Pull quantifiers to the front (existentials before universals):

$$\exists v\, \exists w\, \forall u\, (P(u) \land Q(v) \land (\neg R(w) \land S(w)))$$

So the prenex form is: $\exists v\, \exists w\, \forall u\, (P(u) \land Q(v) \land \neg R(w) \land S(w))$.

c)

$$\begin{aligned} & \forall x\, P(x) \leftrightarrow \exists x\, Q(x) \\ \equiv\ & (\forall x\, P(x) \rightarrow \exists x\, Q(x)) \land (\exists x\, Q(x) \rightarrow \forall x\, P(x)) \\ \equiv\ & (\neg \forall x\, P(x) \lor \exists x\, Q(x)) \land (\neg \exists x\, Q(x) \lor \forall x\, P(x)) \\ \equiv\ & (\exists x\, \neg P(x) \lor \exists x\, Q(x)) \land (\forall x\, \neg Q(x) \lor \forall x\, P(x)) \end{aligned}$$

For the first disjunction: $\exists x\, \neg P(x) \lor \exists x\, Q(x) \equiv \exists x\, (\neg P(x) \lor Q(x))$. For the second disjunction: $\forall x\, \neg Q(x) \lor \forall x\, P(x)$ cannot be combined directly because the quantifiers are over the same variable. We rename: Let $\forall x\, \neg Q(x)$ become $\forall y\, \neg Q(y)$ and $\forall x\, P(x)$ remain as is. Then we have:

$$\exists x\, (\neg P(x) \lor Q(x)) \land (\forall y\, \neg Q(y) \lor \forall x\, P(x))$$

Now, we can write the conjunction as:

$$\exists x\, (\neg P(x) \lor Q(x)) \land \forall y\, \forall x\, (\neg Q(y) \lor P(x))$$

Pull quantifiers to the front (existential first, then universals):

$$\exists x\, \forall y\, \forall z\, ((\neg P(x) \lor Q(x)) \land (\neg Q(y) \lor P(z)))$$

So the prenex form is: $\exists x\, \forall y\, \forall z\, ((\neg P(x) \lor Q(x)) \land (\neg Q(y) \lor P(z)))$.

Exercise 5

Put the following formulas into prenex and skolem form:

a) $(\exists z\, \forall t\, S(z, t)) \rightarrow (\forall x\, \exists y\, \forall z\, R(x, y, z))$

b) $\forall x\, \exists z\, R(x, z) \leftrightarrow \forall x\, \forall y\, R(x, y)$

c) $\forall x\, \forall y\, (F(x, y) \rightarrow (\exists z\, P(z, y) \land \forall z\, P(z, x)))$

d) $(\forall x\, P(x) \land \forall x\, \forall y\, Q(x, y)) \rightarrow \exists z\, \forall t\, A(z, t)$

Solution

a)

$$
\begin{aligned}
& (\exists z\, \forall t\, S(z,t)) \to (\forall x\, \exists y\, \forall z\, R(x,y,z)) \\
\equiv\ & \neg(\exists z\, \forall t\, S(z,t)) \vee (\forall x\, \exists y\, \forall z\, R(x,y,z)) \\
\equiv\ & \forall z\, \exists t\, \neg S(z,t) \vee \forall x\, \exists y\, \forall z\, R(x,y,z)
\end{aligned}
$$

Rename variables to avoid conflict: In the first part, we have $\forall z\, \exists t\, \neg S(z,t)$. In the second part, the $\forall z$ is in the scope of $\exists y$ and $\forall x$. We rename the z in the second part to u:

$$\forall z\, \exists t\, \neg S(z,t) \vee \forall x\, \exists y\, \forall u\, R(x,y,u)$$

Now, we put into prenex form. Since the first part has $\forall z$ and $\exists t$, and the second has $\forall x$, $\exists y$, and $\forall u$, we can pull all quantifiers to the front. However, note that the existential quantifiers depend on the universals before them. We have:

$$\forall z\, \exists t\, \neg S(z,t) \vee \forall x\, \exists y\, \forall u\, R(x,y,u)$$

We cannot simply reorder the quantifiers because of dependencies. Instead, we use Skolemization to remove existential quantifiers. First, write in prenex form without changing the order:

Let's do step by step:

$$
\begin{aligned}
& \forall z\, \exists t\, \neg S(z,t) \vee \forall x\, \exists y\, \forall u\, R(x,y,u) \\
\equiv\ & \forall z\, \exists t\, \forall x\, \exists y\, \forall u\, (\neg S(z,t) \vee R(x,y,u))
\end{aligned}
$$

Explanation: We can pull the quantifiers out because the variables are distinct and the formula is a disjunction. However, note that the existential quantifiers $\exists t$ and $\exists y$ depend on the universals that come before them. So the order is: $\forall z\, \exists t\, \forall x\, \exists y\, \forall u$.

Now, we Skolemize: Replace t by a Skolem function $f(z)$ and y by a Skolem function $g(z,x)$. Then we get:

$$\forall z\, \forall x\, \forall u\, (\neg S(z, f(z)) \vee R(x, g(z,x), u))$$

b)

$$
\begin{aligned}
&\forall x\, \exists z\, R(x,z) \leftrightarrow \forall x\, \forall y\, R(x,y) \\
\equiv\ &(\forall x\, \exists z\, R(x,z) \to \forall x\, \forall y\, R(x,y)) \land (\forall x\, \forall y\, R(x,y) \to \forall x\, \exists z\, R(x,z))
\end{aligned}
$$

Consider the first part:

$$
\begin{aligned}
&\forall x\, \exists z\, R(x,z) \to \forall x\, \forall y\, R(x,y) \\
\equiv\ &\neg(\forall x\, \exists z\, R(x,z)) \lor \forall x\, \forall y\, R(x,y) \\
\equiv\ &\exists x\, \forall z\, \neg R(x,z) \lor \forall x\, \forall y\, R(x,y)
\end{aligned}
$$

Put into prenex form:

$$\exists x\, \forall z\, \forall x'\, \forall y\, (\neg R(x,z) \lor R(x',y))$$

Rename x in the second part to x' to avoid conflict. Now, Skolemize: Replace the existential x by a constant c. Then:

$$\forall z\, \forall x'\, \forall y\, (\neg R(c,z) \lor R(x',y))$$

So the first part becomes: $\neg R(c,z) \lor R(x',y)$.

Now the second part:

$$
\begin{aligned}
&\forall x\, \forall y\, R(x,y) \to \forall x\, \exists z\, R(x,z) \\
\equiv\ &\neg(\forall x\, \forall y\, R(x,y)) \lor \forall x\, \exists z\, R(x,z) \\
\equiv\ &\exists x\, \exists y\, \neg R(x,y) \lor \forall x\, \exists z\, R(x,z)
\end{aligned}
$$

Put into prenex form:

$$\exists x\, \exists y\, \forall x'\, \exists z\, (\neg R(x,y) \lor R(x',z))$$

Skolemize: Replace x by constant a, y by constant b, and z by a Skolem function $h(x')$. Then:

$$\forall x'\, (\neg R(a,b) \lor R(x', h(x')))$$

So the second part becomes: $\neg R(a,b) \lor R(x', h(x'))$.

c)

$$\begin{aligned}&\forall x\,\forall y\,(F(x,y)\to(\exists z\,P(z,y)\wedge\forall z\,P(z,x)))\\ \equiv\ &\forall x\,\forall y\,(\neg F(x,y)\vee(\exists z\,P(z,y)\wedge\forall u\,P(u,x)))\end{aligned}$$

We renamed the second z to u to avoid conflict. Now, distribute the disjunction over the conjunction:

$$\forall x\,\forall y\,((\neg F(x,y)\vee\exists z\,P(z,y))\wedge(\neg F(x,y)\vee\forall u\,P(u,x)))$$

Now, put each part into prenex form separately.

For the first conjunct: $\forall x\,\forall y\,(\neg F(x,y)\vee\exists z\,P(z,y))\equiv\forall x\,\forall y\,\exists z\,(\neg F(x,y)\vee P(z,y))$.

For the second conjunct: $\forall x\,\forall y\,(\neg F(x,y)\vee\forall u\,P(u,x))\equiv\forall x\,\forall y\,\forall u\,(\neg F(x,y)\vee P(u,x))$.

Now, Skolemize the first conjunct: we have $\exists z$ depending on x and y, so introduce a Skolem function $g(x,y)$:

$$\forall x\,\forall y\,(\neg F(x,y)\vee P(g(x,y),y))$$

The second conjunct is already without existentials.

d)

$$\begin{aligned}&(\forall x\,P(x)\wedge\forall x\,\forall y\,Q(x,y))\to\exists z\,\forall t\,A(z,t)\\ \equiv\ &\neg(\forall x\,P(x)\wedge\forall x\,\forall y\,Q(x,y))\vee\exists z\,\forall t\,A(z,t)\\ \equiv\ &(\exists x\,\neg P(x)\vee\exists x\,\exists y\,\neg Q(x,y))\vee\exists z\,\forall t\,A(z,t)\end{aligned}$$

Put into prenex form: combine the existential quantifiers. We have three existential quantifiers and then a universal one. We can write:

$$\exists x\,\exists x'\,\exists y\,\exists z\,\forall t\,(\neg P(x)\vee\neg Q(x',y)\vee A(z,t))$$

Now, Skolemize: Replace x by constant c, x' by constant d, y by constant e, and z by constant a (since z does not depend on any universal variable). Then we get:

$$\forall t\,(\neg P(c)\vee\neg Q(d,e)\vee A(a,t))$$