Resolution of Systems of Linear Equations
Via Direct Methods

1.   The system $Ax = b$ in matrix representation:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n-1}x_{n-1} & + & a_{1n}x_n & & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n-1}x_{n-1} & + & a_{2n}x_n & & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & \dots & + & a_{3n-1}x_{n-1} & + & a_{3n}x_n & = & b_3 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & + & \vdots & & \vdots \\ a_{n-11}x_1 & + & a_{n-12}x_2 & + & \dots & + & a_{n-1n-1}x_{n-1} & + & a_{n-1n}x_n & & b_{n-1} \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn-1}x_{n-1} & + & a_{nn}x_n & & b_n \end{cases}$$

2.   The system's matrix A is

- Not a lower triangular matrix

- Not an upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{bmatrix} \cdot \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} = \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}$$

A method for solving a linear system is termed direct if the solution can be obtained in a finite number of operations.

Direct Methods
Inverse of the System Matrix

$$X = A^{-1}b,$$

This method requires $T_{inverse} = n!\,(n^2 + n + 1) + 3n^2 - 1$ elementary operations.

$A \in M_n(\mathbb{K})$	Number of Elementary Operations
n=3	$T_{inverse} = 104$
n=4	$T_{inverse} = 551$
n=5	$T_{inverse} = 3790$
n=10	$T_{inverse} \sim 4.\,10^8$

Direct Methods
Cramer's Rule

$x_i = \dfrac{\det(A_{i \to b})}{\det(A)}$ where $A_{i \to b}$ is the matrix formed by replacing the i-th column of A by the vector b.

$$x_i = \frac{1}{\det A} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,i-1} & \mathbf{b_1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,i-1} & \mathbf{b_2} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3\,i-1} & \mathbf{b_3} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & & \vdots & \cdots & \vdots \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,i-1} & \mathbf{b_{n-1}} & \cdots & a_{n-1n} \\ a_{n1} & a_{n2} & \cdots & a_{n\,i-1} & \mathbf{b_n} & \cdots & a_{nn} \end{vmatrix}$$

Example

$$\begin{bmatrix} 5 & 4 & 7 \\ 2 & 4 & 8 \\ 3 & 6 & 9 \end{bmatrix} \quad X = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \qquad \det D = -36$$

$$x_1 = \frac{-1}{36} \begin{vmatrix} 10 & 4 & 7 \\ 20 & 4 & 8 \\ 30 & 6 & 9 \end{vmatrix} \qquad x_2 = \frac{-1}{36} \begin{vmatrix} 5 & 10 & 7 \\ 2 & 20 & 8 \\ 3 & 30 & 9 \end{vmatrix} \qquad x_3 = \frac{-1}{36} \begin{vmatrix} 5 & 4 & 10 \\ 2 & 4 & 20 \\ 3 & 6 & 30 \end{vmatrix}$$

$$x_1 = -3.3333 \qquad x_2 = 6.6667 \qquad x_3 = 0$$

Direct Methods

Cramer's Rule

Solving the system using Cramer's Rule requires a total of:

$$T_{Cramer} = (n + 1)^2. n! - 1 \text{ elementary operations}$$

$A \in M_n(\mathbb{K})$	Number of Elementary Operations
n=3	$T_{Cramer} = 95$
n=4	$T_{Cramer} = 599$
n=5	$T_{Cramer} = 4319$
n=10	$T_{Cramer} \sim 4.10^8$

This method becomes very slow in terms of execution time (computation) as soon as n exceeds 4. ➜ It is therefore not more advantageous than computing the inverse.
.

Direct Methods : Gaussian Elimination Method

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \;\; + \;\; \vdots \;\; + \ddots + \;\; \vdots \;\; &= \vdots \\
a_{n-1}x_1 + a_{n-1\,2}x_2 + \ldots + a_{n-1n}x_n &= b_{n-1} \\
a_{n1}x_1 + a_{n\,2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1,n-1} & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2,n-1} & a_{2n} \\
a_{31} & a_{32} & \ldots & a_{3,n-1} & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} & a_{n-1\,2} & \ldots & a_{n-1,n-1} & a_{n-1n} \\
a_{n1} & a_{n\,2} & \ldots & a_{n-1\,n} & a_{nn}
\end{bmatrix}
\times
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n
\end{bmatrix}
$$

Objective : A x = b

→

R x = c

Where R is an upper triangular matrix.

$$
\begin{bmatrix}
r_{11} & r_{12} & \ldots & r_{1,n-1} & r_{1n} \\
0 & r_{22} & \ldots & r_{2,n-1} & r_{2n} \\
0 & 0 & \ddots & r_{3,n-1} & r_{3,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & r_{n-1,n-1} & r_{n-1n} \\
0 & 0 & \ldots & 0 & r_{nn}
\end{bmatrix}
\times
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n
\end{bmatrix}
$$

Direct Methods : Gaussian Elimination Method

We denote the initial system as $A^{(0)}x = b^{(0)}$

(Step 1) $\rightarrow$ Result : $A^{(1)}x = b^{(1)}$

$\vdots$

(Step i) $\rightarrow$ Result : $A^{(i)}x = b^{(i)}$

(Final Step (n-1)) $\rightarrow$ Result : $\rightarrow$ $\mathbf{A^{(n-1)}x = b^{(n-1)}}$

The process uses the **augmented matrix** $[A \mid b]$.

Example : Solve the following system:

$$
\begin{array}{ccccccccc}
x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & & 11 \\
2x_1 & + & 3x_2 & + & 4x_3 & + & x_4 & & 12 \\
3x_1 & + & 4x_2 & + & x_3 & + & 2x_4 & = & 13 \\
4x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & & 14
\end{array}
$$

$A^{(0)}x = b^{(0)}$ **Augmented Matrix**

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 11 \\
2 & 3 & 4 & 1 & 12 \\
3 & 4 & 1 & 2 & 13 \\
4 & 1 & 2 & 3 & 14
\end{bmatrix}
$$

Direct Methods : Gaussian Elimination Method

Step 1: <u>Work on the 1st column (j=1)</u> of $A^{(0)}x = b^{(0)}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 11 \\ 2 & 3 & 4 & 1 & 12 \\ 3 & 4 & 1 & 2 & 13 \\ 4 & 1 & 2 & 3 & 14 \end{array}\right]$$

<u>**Objective**</u> Eliminate the coefficients $a_{i1}^{(1)}$ of $A^{(0)}$, $i = 2:n$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 11 \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \end{array}\right]$$

Method ?

$$\begin{cases} x_1 +2x_2+3x_3+4x_4 & 11 \\ 2x_1+3x_2+4x_3+ x_4 & 12 \\ 3x_1+4x_2+ x_3 +2x_4 & 13 \\ 4x_1+ x_2 +2x_3+3x_4 & 14 \end{cases} = \begin{matrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \rightarrow \begin{matrix} L_1 \\ L_2 = L_2 - \frac{2}{1}L_1 \\ L_3 = L_3 - \frac{3}{1}L_1 \\ L_4 = L_4 - \frac{4}{1}L_1 \end{matrix}$$

$$A^{(1)}x = b^{(1)} \quad \equiv \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & -2 & -8 & -10 & -20 \\ 0 & -7 & -10 & -13 & -30 \end{array}$$

Step 1:

Work on the 1ˢᵗ column (j=1)   of $A^{(0)}x = b^{(0)}$

Objective Eliminate the coefficients $a_{i1}^{(1)}$ from $A^{(0)}$ $i = 2{:}n$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \end{bmatrix}$$

Method

- Verify the pivot $a_{11}^{(0)} \neq 0$ Here, $a_{11}^{(0)} = 1 \neq 0$
- For each row L_i where $i = 2 : n$,

    subtract the first row L_1 multiplied by $\dfrac{a_{i1}}{a_{11}}$:

for i=2 : $L_2 \leftarrow L_2 - \dfrac{a_{21}^{(0)}}{a_{11}^{(0)}} L_1$ $\rightarrow$ $\begin{matrix} a_{11} & \dots \\ 0 & \dots \\ a_{31} & \dots \\ \dots & \dots \\ a_{n1} & \dots \end{matrix}$

for i=3 : $L_3 \leftarrow L_3 - \dfrac{a_{31}^{(0)}}{a_{11}^{(0)}} L_1$ $\rightarrow$ $\begin{matrix} a_{11} & \dots \\ 0 & \dots \\ 0 & \dots \\ \dots & \dots \\ a_{n1} & \dots \end{matrix}$

$$\text{for} \quad i = n \; : L_n \leftarrow L_n - \frac{a_{n1}^{(0)}}{a_{11}^{(0)}} L_1 \quad \rightarrow \quad \begin{matrix} a_{11} & \dots \\ 0 & \dots \\ 0 & \dots \\ \dots & \dots \\ 0 & \dots \end{matrix}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 & 11 \\ 2x_1 + 3x_2 + 4x_3 + x_4 \\ 3x_1 + 4x_2 + x_3 + 2x_4 \\ 4x_1 + x_2 + 2x_3 + 3x_4 & 14 \end{cases} = \begin{matrix} 11 \\ 12 \\ 13 \\ 14 \end{matrix} \rightarrow \begin{matrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \rightarrow \begin{matrix} L_1 \\ L_2 = L_2 - \frac{2}{1} L_1 \\ L_3 = L_3 - \frac{3}{1} L_1 \\ L_4 = L_4 - \frac{4}{1} L_1 \end{matrix}$$

Computation

$$\text{Matrix representation} : \begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 2 & 3 & 4 & 1 & 12 \\ 3 & 4 & 1 & 2 & 13 \\ 4 & 1 & 2 & 3 & 14 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ (1\ 2\ 3\ 4\ 11) \times 2/1 \\ (1\ 2\ 3\ 4\ 11) \times 3/1 \\ (1\ 2\ 3\ 4\ 11) \times 4/1 \end{bmatrix}$$

$$A^{(1)}x = b^{(1)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 2 & 3 & 4 & 1 & 12 \\ 3 & 4 & 1 & 2 & 13 \\ 4 & 1 & 2 & 3 & 14 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 8 & 22 \\ 3 & 6 & 9 & 12 & 33 \\ 4 & 8 & 12 & 16 & 44 \end{bmatrix} = \begin{matrix} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & -2 & -8 & -10 & -20 \\ 0 & -7 & -10 & -13 & -30 \end{matrix}$$

Step 2:

Work on the 2nd column (j=2)   of $A^{(1)}x = b^{(1)}$

Objective Eliminate the coefficients $a_{i2}^{(2)}$ from $A^{(1)}$ $i = 3:n$ (3,4)

$$\begin{matrix} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & -2 & -8 & -10 & -20 \\ 0 & -7 & -10 & -13 & -30 \end{matrix}$$

Method :

1. Verify the pivot : $a_{22}^{(1)} \neq 0$

2. For each row L_i, n=3 :4   subtract the second row L_2 multiplied by $\dfrac{a_{i2}}{a_{22}}$

for i=3: $L_3 \leftarrow L_3 - \dfrac{a_{32}^{(1)}}{a_{22}^{(1)}} L_2$

$$\begin{matrix} a_{11} & a_{12} & \dots \\ 0 & a_{22} & \dots \\ 0 & 0 & \dots \\ 0 & a_{42} & \dots \\ \dots & \dots & \dots \\ 0 & a_{n2} & \dots \end{matrix}$$

$$\text{for } i=4 : L_4 \leftarrow L_4 - \frac{a_{42}^{(1)}}{a_{22}^{(1)}} L_2 \qquad \begin{array}{ccc} a_{11} & a_{12} & \dots \\ 0 & a_{22} & \dots \\ 0 & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & a_{n2} & \dots \end{array}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 \\ \boxed{-x_2} - 2x_3 - 7x_4 \\ -2x_2 - 8x_3 - 10x_4 \\ -7x_2 - 10x_3 - 13x_4 \end{cases} = \begin{matrix} 11 \\ -10 \\ -20 \\ -30 \end{matrix} \quad \begin{matrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \rightarrow \begin{matrix} L_1 \\ L_2 \\ L_3 = L_3 - \frac{(-2)}{(-1)} L_2 \\ L_4 = L_4 - \frac{(-7)}{(-1)} L_2 \end{matrix}$$

Computation :

Matrix representation : $\begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 0 & \boxed{-1} & -2 & -7 & -10 \\ 0 & -2 & -8 & -10 & -20 \\ 0 & -7 & -10 & -13 & -30 \end{bmatrix} - \begin{bmatrix} / & / & / & / & / \\ / & / & / & / & / \\ (0 & -1 & -2 & -7 & -10) & \times & -2/\boxed{-1} \\ (0 & -1 & -2 & -7 & -10) & \times & -7/\boxed{-1} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & -2 & -8 & -10 & -20 \\ 0 & -7 & -10 & -13 & -30 \end{bmatrix} - \begin{bmatrix} / & / & / & / & / \\ / & / & / & / & / \\ (0 & -2 & -4 & -14 & -20) \\ (0 & -7 & -14 & -49 & -70) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 4 & 36 & 40 \end{bmatrix} = A^{(2)} x = b^{(2)}$$

Step (n-1): <u>Work on the (n-1)th column (j=n-1)</u> of $A^{(n-2)}x = b^{(n-2)}$ $\equiv$

Step (3): <u>Work on the (3)rd column (j=3)</u> of $A^{(2)}x = b^{(2)}$

<u>Objective</u> Eliminate the coefficients $a_{i3}^{(3)}(a_{i,n-1}^{(n-1)})$ from $A^{(2)}$ $i = n = 4$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & -2 & -8 & -10 & -20 \\ 0 & -7 & -10 & -13 & -30 \end{array}$$

<u>Method :</u>

1. Verify the pivot $a_{n-1,n-1}^{(n-2)} \neq 0$

2. For the line L_i, where $i = j+1 : n$, subtract the line L_{n-1} multiplied by $\dfrac{a_{i,n-1}}{a_{n-1,n-1}}$

Pour i=n : $L_n \leftarrow L_n - \dfrac{a_{i,n-1}^{(n-2)}}{a_{n-1,n-1}^{(n-2)}} L_{n-1}$

$A^{(n-1)}x = b^{(n-1)}$

$$\begin{array}{cccccc} r_{11} & r_{12} & \cdots & r_{1,n-1} & r_{1n} & c_1 \\ 0 & r_{22} & \cdots & r_{2,n-1} & r_{2n} & c_2 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{n-1,n-1} & r_{n-1n} & c_{n-1n} \\ 0 & 0 & \cdots & 0 & r_{nn} & c_{nn} \end{array}$$

Computation

$$
\begin{cases}
x_1 + 2x_2 + 3x_3 + 4x_4 & 11 \\
\quad\;\; -x_2 - 2x_3 - 7x_4 & -10 \\
\quad\quad\quad\; \boxed{-4x_3} + 4x_4 & 0 \\
\quad\quad\quad\quad\; 4x_3 + 36x_4 & 40
\end{cases}
=
\quad
\begin{matrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix}
\rightarrow
\quad
\begin{matrix} L_1 \\ L_2 \\ L_3 \\ L_4 = L_4 - \dfrac{(4)}{(-4)} L_3 \end{matrix}
$$

Matrix representation :

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 11 \\
0 & -1 & -2 & -7 & -10 \\
0 & 0 & \boxed{-4} & +4 & 0 \\
0 & 0 & +4 & +36 & 40
\end{bmatrix}
-
\begin{bmatrix}
/ & / & / & / & / \\
/ & / & / & / & / \\
/ & / & / & / & / \\
(0 & 0 & -4 & +4 & 0) \times \mathbf{4}/\boxed{-4}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
1 & 2 & 3 & 4 & 11 \\
0 & -1 & -2 & -7 & -10 \\
0 & 0 & \boxed{-4} & +4 & 0 \\
0 & 0 & +4 & +36 & 40
\end{bmatrix}
-
\begin{bmatrix}
/ & / & / & / & / \\
/ & / & / & / & / \\
/ & / & / & / & / \\
(0 & 0 & +4 & -4 & 0)
\end{bmatrix}
=
\begin{bmatrix}
1 & 2 & 3 & 4 & 11 \\
\mathbf{0} & -1 & -2 & -7 & -10 \\
\mathbf{0} & \mathbf{0} & -4 & 4 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & 40 & 40
\end{bmatrix}
= A^{(3)} x = b^{(3)}
$$

Final step : Back Substitution

1. The determinant of the matrix $A^{(3)}$ is verified from the triangular form :

$$\det A^{(3)} = \prod_{j=1}^{j=n} a_{jj}^{(n-1)};$$

$$\det A^{(3)} = (1) \times (-1) \times (-4) \times (40) = 160$$

2. Solve the triangular system $A^{(3)}x = b^{(3)}$ using back substitution:

$$
\begin{array}{llll}
40x_4 & = 40 & \Rightarrow x_4 = 1 \\
-4x_3 + 4(1) & = 0 & \Rightarrow x_3 = 1 \\
-x_2 - 2(1) - 7(1) & = -10 & \Rightarrow x_2 = 1 \\
x_1 + 2(1) + 3(1) + 4(1) & = 11 & \Rightarrow x_1 = 2
\end{array}
$$

Solution:

x4= 1,
x3= 1,
x2= 1 ,
x1= 2

Gaussian Elimination Method (Summary)

The process consists of two phases: (1) Triangularization (Forward Elimination), (2) Back Substitution.

Triangularization (Forward Elimination)

Start with A, the systems' matrix of order n. The same operations are performed on the right side vector b.

- There are $n - 1$ steps: initial system (0) $\rightarrow$ final system (n-1), which is upper triangular.

- At step j, we eliminate the coefficients below the diagonal in column j of the previous system $A^{(j-1)}x = b^{(j-1)}$.

How? For each row $i > j$, subtract row j multiplied by $\dfrac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}}$

- $j < i \leq n$ we have $a_{ij}^{(j)} = 0$

- $j \leq k \leq n$ we have $a_{ik}^{(j)} = a_{ik}^{(j-1)} - \dfrac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}} a_{jk}^{(j-1)}$ and $b_i^{(j)} = b_i^{(j-1)} - \dfrac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}} b_j^{(j-1)}$

<u>**Pseudo code**</u>

For   j = 1 to n-1
   if pivot $a_{jj}^{(j-1)} \neq 0$
      for i = j+1 to n
$$L_i \leftarrow L_i - \frac{a_{i,j}^{(j-1)}}{a_{j,j}^{(j-1)}} L_j$$

Gaussian Elimination code

```matlab
function [x,det] = gauss(A,b)    % et calcul de solution et déterminant
n = length(b);
if (size(b,2) == 1 & isvector(b) & size(A)==n)
% vérifier que b est un vecteur de taille nx1 et A matrice de taille nxn
%******************************* début phase d'élimination
for j = 1:n-1       %parcours des colonnes
    if A(j,j)~= 0  % vérifier  pivot
    for i= j+1:n       %parcours des lignes
        coef= A(i,j)/A(j,j); %constante
        A(i,j:n) = A(i,j:n) – coef * A(j,j:n); % Li =  Li  - Lj* coef
        b(i)= b(i) - coef*b(j);   % cohérence du système (matrice augmentée)
    end
  end
end
end
 det = prod(diag(A));  %vérification obligatoire
 %***************** phase de résolution du syst triang sup (par vectorisation)
for k = n:-1:1
    b(k) = (b(k) - A(k,k+1:n)*b(k+1:n))/A(k,k);
end,
x = b;
```

Gaussian Elimination Method (REMARKS)

Remark 1

The solution of a linear system is unchanged when:

- Two rows are permuted.

- Two columns are permuted (note: this permutes the solution variables).

- A row is multiplied by a non-zero scalar.

- A multiple of one row is added to another row.

Remarque 2

Problem : The basic Gaussian elimination method fails if a pivot $a_{k+1,k+1}^{(k)} = 0$

Solution: Use **partial pivoting**: swap row $L_{k+1}^{(k)}$ with a row $L_i^{(k)}$ (where $i > k$) such that $| a_{i,k+1}^{(k)} |$ is maximized.

Example :

Initial augmented matrix $[A^{(0)} \mid b^{(0)}]$

$$\begin{bmatrix} 0 & -1 & 2 & 13 & 5 \\ 1 & -2 & 3 & 17 & 4 \\ -1 & 3 & -3 & -20 & -1 \\ 0 & -5 & -1 & 4 & 1 \end{bmatrix} \quad \text{has} \quad [A^{(0)}b^{(0)}] \quad a_{11} = 0 \rightarrow \text{Swap} \quad L_2^{(0)} \leftrightarrow L_1^{(0)}$$

Résult :

$$\begin{bmatrix} 1 & -2 & 3 & 17 & 4 \\ 0 & -1 & 2 & 13 & 5 \\ -1 & 3 & -3 & -20 & -1 \\ 0 & -5 & -1 & 4 & 1 \end{bmatrix} \quad [A^{(0)}b^{(0)}]$$

Proceed with elimination $\quad L_i \leftarrow L_i + \frac{1}{1} L_1 \quad \begin{bmatrix} 1 & -2 & 3 & 17 & 4 \\ 0 & -1 & 2 & 13 & 5 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -5 & -1 & 4 & 1 \end{bmatrix} \quad [A^{(1)}b^{(1)}]$

.....

Remark 3

The Gaussian elimination method solves a linear system using an estimated number of elementary operations:

$T_{Gauss} = \frac{2n^3}{3} + O(n^2) \quad$ or more precisely $= (4n^3 + 9n^2 - 7n)/6,$

This demonstrates the massive improvement this method offers over inverse and Cramer methods.

$A \in M_n(\mathbb{K})$	Number of Operations
n=3	$T_{Gauss} = 28$
n=4	$T_{Gauss} = 62$
n=5	$T_{Gauss} = 115$
n=10	$T_{Gauss} = 805$

Remark 4

Using Gaussian elimination, the determinant of matrix A is calculated as follows:

$$\det A = (-1)^p \prod_{k=1}^{k=n} a_{kk}^{(k)} ;$$

where p is the number of row permutations performed during the elimination process.