Direct Methods for Solving Systems of Linear Equations

Introduction

One of the most frequent and fundamental applications of matrices is the representation and solution of systems of linear equations. This matrix-based formulation provides a compact, efficient, and systematic framework for analyzing and solving linear problems across science, engineering, and applied mathematics.

Direct Methods for Solving Systems of Linear Equations

Introduction

Consider a system of three equations in three unknowns of the form:

$$\begin{cases} a_{11} & x_1 & + & a_{12} & x_2 & + & a_{13} & x_3 & = & b_1 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & a_{23} & x_3 & = & b_2 \\ a_{31} & x_1 & + & a_{32} & x_2 & + & a_{33} & x_3 & = & b_3 \end{cases}$$

Traditional Substitution Method for solving this system :

1. Isolate one variable (x_1)

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

2. Substitute into the other equations to obtain a 2×2 system in x_2 and x_3.

3. Solve the reduced 2×2 system : Compute x_2 and x_3 from the new system.

4. Back-substitute x_2 and x_3 into the expression of x_1

Direct Methods for Solving Systems of Linear Equations

Introduction

The traditional substitution method is tedious and does not scale to larger systems.

Using matrices allows a compact, systematic, and efficient way to represent and solve linear systems.

Link Between Linear Systems and Matrices

This system $\begin{cases} a_{11} & x_1 & + & a_{12} & x_2 & + & a_{13} & x_3 & = & b_1 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & a_{23} & x_3 & = & b_2 \\ a_{31} & x_1 & + & a_{32} & x_2 & + & a_{33} & x_3 & = & b_3 \end{cases}$

admits the following matrix representation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Equivalent to $\mathrm{A} \;.\; \boldsymbol{x} = \boldsymbol{b}$

Link Between Linear Systems and Matrices

- The matrix $A \in M_n(\mathbb{K})$ is called **the coefficient matrix of the system**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- The vector $\mathrm{b} \in \mathbb{K}^n$ is called the **right–hand side vector**. $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

- The vector $x \in \mathbb{K}^n$ is the **vector of unknowns**. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Link Between Linear Systems and Matrices

$$\begin{cases} 5x_1 & + & 4x_2 & + & 7x_3 & = & 10 \\ 2x_1 & + & 4x_2 & + & 8x_3 & = & 20 \\ 3x_1 & + & 6x_2 & + & 9x_3 & = & 30 \end{cases} \quad \equiv \quad \begin{bmatrix} 5 & 4 & 7 \\ 2 & 4 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

The coefficient matrix of the system $A = \begin{bmatrix} 5 & 4 & 7 \\ 2 & 4 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

vector of unknowns $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **right−hand side vector** $\mathrm{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$

Matrix-Based Solution of a System of Linear Equations

Once the matrix representation has been properly established, how can we solve these systems?

Matrix-Based Solution of a System of Linear Equations

PROBLEM :

$$\begin{bmatrix} 5 & 4 & 7 \\ 2 & 4 & 8 \\ 3 & 6 & 9 \end{bmatrix} X = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \quad \equiv \quad A\,x \,=\, b, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ????$$

SOLUTION :

$$x = \begin{bmatrix} 0.3333 & -0.1667 & -0.1111 \\ -0.1667 & -0.6667 & 0.7222 \\ 0.0000 & 0.5000 & -0.3333 \end{bmatrix} . \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$x = A^{-1} . b$$

$$= \begin{bmatrix} -3.3333 \\ 6.6666 \\ 0 \end{bmatrix}$$

$X = \text{inv}(A) * b$ %via MATLAB

Matrix-Based Solution of a System of Linear Equations

$$A\,x = b \qquad \Longrightarrow \qquad x = A^{-1}\,.\,b$$

Interpret the number of solutions in terms of the determinant :

det(A)≠0:

The matrix is invertible, which means that the system has a unique solution.

Otherwise,

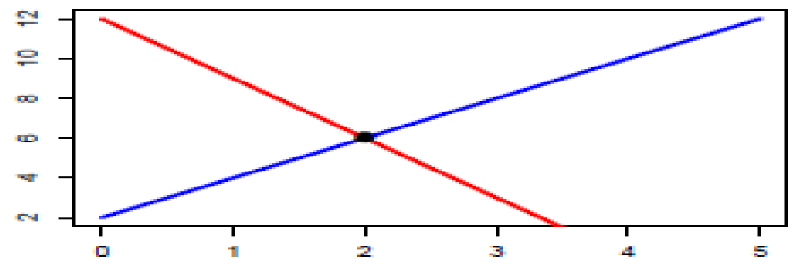
det(A)=0:

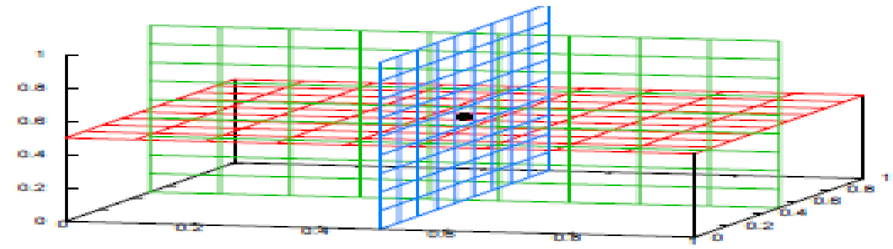
- The system has no solution.
- The system admits infinitely many solutions.

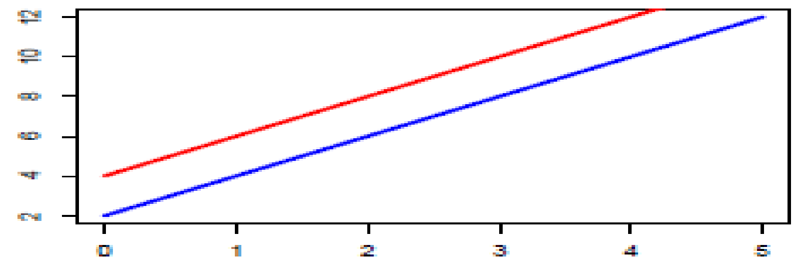
Matrix-Based Solution of a System of Linear Equations

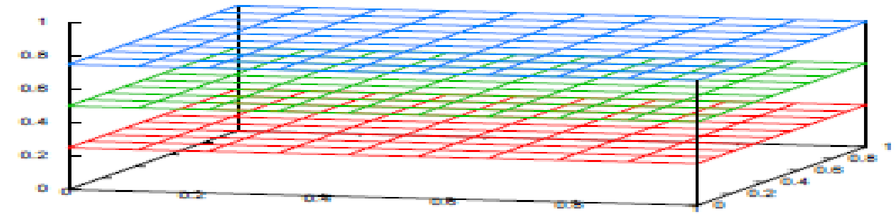
$$\begin{cases} x - 2y = 0 \\ x + y = 3 \end{cases}$$

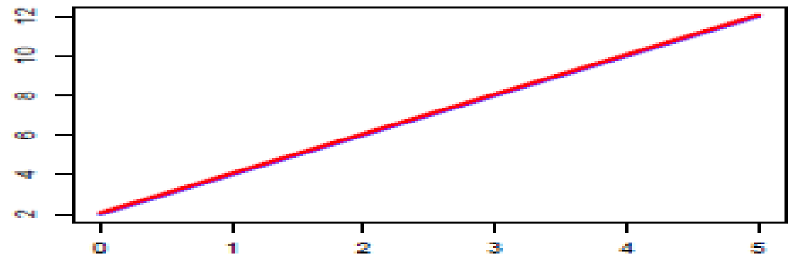
$$\begin{cases} 5x_1 + 4x_2 + 7x_3 = 10 \\ 2x_1 + 4x_2 + 8x_3 = 20 \\ 3x_1 + 6x_2 + 9x_3 = 30 \end{cases}$$

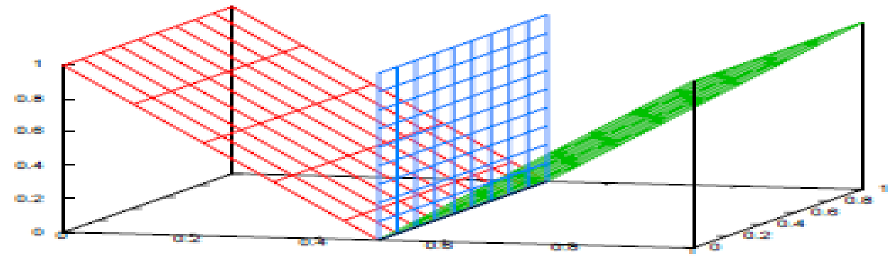












Solving a Lower-Triangular System

Solving a Lower-Triangular System

Given the linear system $Ax = b$ with $A \in \mathbb{R}^{4\times 4}$ a lower-triangular matrix and $x, b \in \mathbb{R}^4$.

$$\begin{cases} -3x_1 = 10 \\ 2x_1 + 4x_2 = -4 \\ x_1 - 3x_2 + 2x_3 = 8 \\ -4x_1 + 5x_2 + 6x_3 + x_4 = -3 \end{cases}$$

$$\begin{cases} -3x_1 & & & & = 10 \\ 2x_1 & + 4x_2 & & & = -4 \\ x_1 & - 3x_2 & + 2x_3 & & = 8 \\ -4x_1 & + 5x_2 & + 6x_3 & + x_4 & = -3 \end{cases}$$

$$A\,x = b$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ -4 & 5 & 6 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \\ 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

A is **lower triangular**,
$a_{ij} = 0$ whenever $i < j$.

Solving a Lower-Triangular System

$$\begin{cases} -3x_1 & & & & = 10 \\ 2x_1 + 4x_2 & & & & = -4 \\ x_1 - 3x_2 + 2x_3 & & & & = 8 \\ -4x_1 + 5x_2 + 6x_3 + x_4 & & & & = -3 \end{cases} \equiv \begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ -4 & 5 & 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \\ 8 \\ -3 \end{bmatrix}$$

$$x_1 = \frac{-10}{3}$$

for i = 1, $x_1 = \frac{10}{-3}$

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{1}{4}(-4 - 2x_1)$$

for i = 2, $x_2 = \frac{1}{4}(-4 - 2x_1)$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1)$$

$$x_3 = \frac{1}{2}(8 - 1.x_1 + 3.x_2)$$

for i = 3, $x_3 = \frac{1}{2}(8 - 1.x_1 + 3.x_2)$

$$x_3 = \frac{1}{a_{33}}(b_3 - (a_{31}x_1 + a_{32}x_2))$$

$$x_4 = \frac{1}{1}(-3 + 4.x_1 - 5x_2 - 6x_3)$$

for i = 4, $x_4 = \frac{1}{1}(-3 - 3 + 4.x_1 - 5x_2 - 6x_3)$

$$x_4 = \frac{1}{a_{44}}(b_4 - (a_{41}x_1 + a_{42}x_2 + a_{43}x_3))$$

Solving a Lower-Triangular System

$$\begin{cases} a_{11}\boldsymbol{x_1} & + & \mathbf{0} & + & \mathbf{0} & + & \cdots & \mathbf{0} & & b_1 \\ a_{21}\boldsymbol{x_1} & + & a_{22}\boldsymbol{x_2} & + & \mathbf{0} & + & \cdots & \mathbf{0} & & b_2 \\ a_{31}\boldsymbol{x_1} & + & a_{32}\boldsymbol{x_2} & + & a_{33}\boldsymbol{x_3} & + & \cdots & \mathbf{0} & = & b_3 \\ \vdots & + & \vdots & + & \vdots & + & \ddots & \vdots & & \vdots \\ a_{n1}\boldsymbol{x_1} & + & a_{n2}\boldsymbol{x_2} & + & \cdots & + & \cdots & a_{nn}\boldsymbol{x_n} & & b_n \end{cases}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \iddots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$\text{for } i = n \quad , x_n = \frac{1}{a_{nn}} \left(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nn-1}x_{n-1} \right)$$

Solving a Lower-Triangular System

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j\right), \ i = 2, \ldots, n.$$

Forward substitution (method of descent)

- Compute x_1
- Compute x_i (rows $i = 2:n$)
  - process columns j :
  - $i < j, \quad a_{ij} = 0 \rightarrow \quad /$
  - $i \geq j \quad a_{ij}$ used to compute x_i
    - $i = j \rightarrow a_{ii}$ is the **denominator** in the expression for x_i
    - $i > j \rightarrow \sum_{j=1}^{i-1} a_{ij} \cdot x_j$

      $(b_i - \sum_{j=1}^{i-1} a_{ij} \cdot x_j) / a_{ii}$

Solving a Lower-Triangular System

Forward substitution (method of descent)

```
function[x]=descente1(A,b)
x(1)=b(1)/A(1,1)
for i=2:size(A,1)
    S=0
    for j=1:(i-1)
        S=S+A(i,j)*x(j)
    end
    x(i)=(b(i)-S)/A(i,i)
end
end
```

```
>>A=[-3 0 0 0;2 4 0  0;1 -3 2 0;-4 5 6 1] ;
>>B=[10;-4;8;-3]

>> dec_lig(A,B), %X=X'
    ans =     -10/3          2/3          20/3          -179/3
```

Solving a Lower-Triangular System

Forward substitution (method of descent)

Remarks:

- Forward substitution requires about $\frac{n(n-1)}{2}$ multiplications and the same order of additions → $O(n^2)$operations.
- Diagonal entries a_{ii}must be non zero.
- Forward substitution is stable and efficient for triangular systems; it is the standard step after Gauss and LU methods.

Solving a Lower-Triangular System

Forward substitution (method of descent)

```
function[x]=dec_lig(A,b)
for i=1:size(A,1)
    x(i)=b(i)
    for j=1:(i-1)
        x(i)=x(i)-A(i,j)*x(j)
    end
    x(i)=x(i)/A(i,i)
end
end
```

```
>>A=[-3 0 0 0;2 4 0  0;1 -3 2 0;-4 5 6 1] ;
>>B=[10;-4;8;-3]
>> X=dec_lig(A,B) %X=X'
X =     -10/3          2/3          20/3          -179/3
```

Solving a Upper-Triangular System

Solving a Upper-Triangular System

Consider the following system to be solved

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n-1}x_{n-1} & + & a_{1n}x_n & & b_1 \\ 0 & + & a_{22}x_2 & + & \dots & + & a_{2n-1}x_{n-1} & + & a_{2n}x_n & & b_2 \\ 0 & + & 0 & + & \dots & + & a_{3n-1}x_{n-1} & + & a_{3n}x_n & = & b_3 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & + & \vdots & & \vdots \\ 0 & + & 0 & + & 0 & + & a_{n-1n-1}x_{n-1} & + & a_{n-1n}x_n & & b_{n-1} \\ 0 & + & 0 & + & 0 & + & 0 & + & a_{nn}x_n & & b_n \end{cases}$$

(Example: a system with 4 unknowns)

$$\begin{cases} 3x_1 & + & 5x_2 & - & 6x_3 & + & x_4 & = 10 \\ & - & 4x_2 & + & 2x_3 & - & 3x_4 & = -4 \\ & & & & 2x_3 & + & 0x_4 & = \ \ 8 \\ & & & & & & 3x_4 & = -3 \end{cases}$$

Solving a Upper-Triangular System

Example:

$$\begin{cases} 3x_1 + 5x_2 - 6x_3 + x_4 = 10 \\ \quad - 4x_2 + 2x_3 - 3x_4 = -4 \\ \quad\quad 2x_3 + 0x_4 = 8 \\ \quad\quad\quad 3x_4 = -3 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = \frac{1}{3}\left(10 - \left(5x_2 - 6x_3 + x_4\right)\right) = \frac{65}{12} \\ x_2 = \frac{1}{-4}\left(-4 - \left(2 \times 4 - 3(-1)\right)\right) = \frac{15}{4} \\ x_3 = \frac{1}{2}\left(8 - 0(-1)\right) = 4 \\ x_4 = \frac{-3}{3} = -1 \end{cases}$$

Solving a Upper-Triangular System

$$\begin{cases} a_{11} & x_1 & + & a_{12} & x_2 & + & a_{13} & x_3 & = & b_1 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & a_{23} & x_3 & = & b_2 \\ a_{31} & x_1 & + & a_{32} & x_2 & + & a_{33} & x_3 & = & b_3 \\ a_{31} & x_1 & + & a_{32} & x_2 & + & a_{33} & x_3 & = & b_4 \end{cases}$$

with a **upper triangular matrix.**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$x_4 = \frac{b_4}{a_{44}}$$

$$x_3 = \frac{1}{a_{33}} (b_3 - a_{34}x_4)$$

$$x_2 = \frac{1}{a_{22}} (b_2 - (a_{23}x_3 + a_{24}x_4))$$

$$x_1 = \frac{1}{a_{11}} (b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4))$$

Solving a Upper-Triangular System

Back Substitution Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- Compute x_n
- Compute x_i $(i = (n-1): -1: 1)$
  - Iterate over colonnes j:
    - $i > j, \quad a_{ij} = 0 \rightarrow \quad /$
    - $i < j$ compute x_i
      using $\sum_{j=i+1}^{n} a_{ij} . x_j$
      or $b_i - \sum_{j=i+1}^{n} a_{ij} . x_j$
    - $i = j \rightarrow a_{ii}$ denominator of the expression

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right), \; i = n-1, \ldots, 1,$$

Solving a Upper-Triangular System

Back Substitution Method

```
function [x]=remonte1(A,b)
n=size(A,1)
x(n)=b(n)/A(n,n)
for i=(n-1):-1:1
    S=0
    for j=(i+1):n%n:-1:i+1
        S=S+A(i,j)*x(j)
    end
    x(i)=(b(i)-S)/A(i,i)
end
end
```

```
>>A=[3 5 -6 1;0 -4 2 -3;0 0 2 0;0 0 0 3],
>>B=[10;-4;8;-3]
>>remonte1(A,B) %X=X'
X = 5.4167    3.7500    4.0000   -1.0000
```

Solving a Upper-Triangular System

Back Substitution Method

```
function[x]=rem_lig(A,b)
for i=(size(A,1)):-1:1
    x(i)=b(i)
    for j=size(A,2): -1:i+1
        x(i)=x(i)-A(i,j)*x(j)
    end
    x(i)=x(i)/A(i,i)
end
end
```

```
>>A=[3 5 -6 1;0 -4 2 -3;0 0 2 0;0 0 0 3],
>>B=[10;-4;8;-3]
>>rem_lig(A,B)    %X=X'
X = 5.4167    3.7500    4.0000   -1.0000
```