5.1. Definition

Torsion (also called pure torsion) is a type of loading such that, in any cross-section of a body (or a part), there exists only a torsional moment M_t

A bar subjected mainly to torsion is called a shaft (Fig. 5.1).

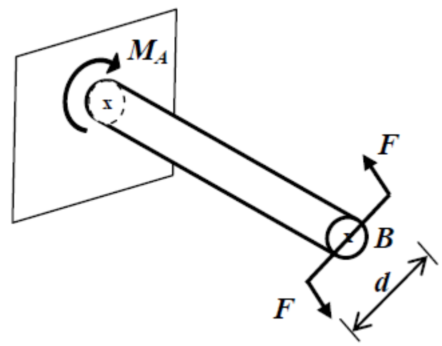


Fig 5.1. Example of a bar subjected to torsion

The equilibrium condition of the above bar is:

$$M_A = M_B = F.d \qquad 5.1$$

As a result, any cross-section of the bar is subjected only to a torsional moment M_t, and this bar is said to be in a state of pure torsion.

5.2. Torsional Moment

5.2.1. Sign Convention

By convention, the Torsional moment is:

- **Positive** ($M_t > 0$) if it acts in the counterclockwise direction for an observer looking at the section.
- **Negative** ($M_t < 0$) if it acts in the clockwise direction for an observer looking at the section.

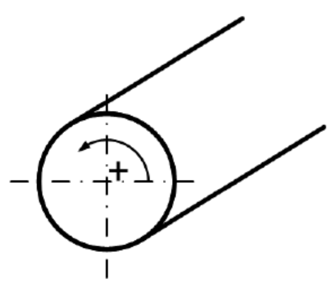


Fig 5.2. Sign Convention

5.2.2. Torsional Moment Diagram

The torsional moment acting at a point of a bar is equal to the algebraic sum of the moments of the external couples applied on either side of the considered part (the part to be considered can be chosen freely).

5.3. Shear Stresses and Torsion Angle

5.3.1. Assumptions

Problems of torsion can be solved using the methods of strength of materials by adopting the following assumptions:

1. The plane cross-section, which is flat and perpendicular to the axis of the member (or element) before loading, remains plane and perpendicular to the axis after loading. This assumption can be experimentally verified for bodies with circular cross-sections but is incorrect for non-circular sections.
2. The diameter of the cross-section before loading remains straight after deformation due to the application of the torsional moment.

5.3.2. Angle of Torsion

Let us consider a bar with a circular cross-section subjected to a constant torsional moment. We cut out (or isolate) an element of length (dx).

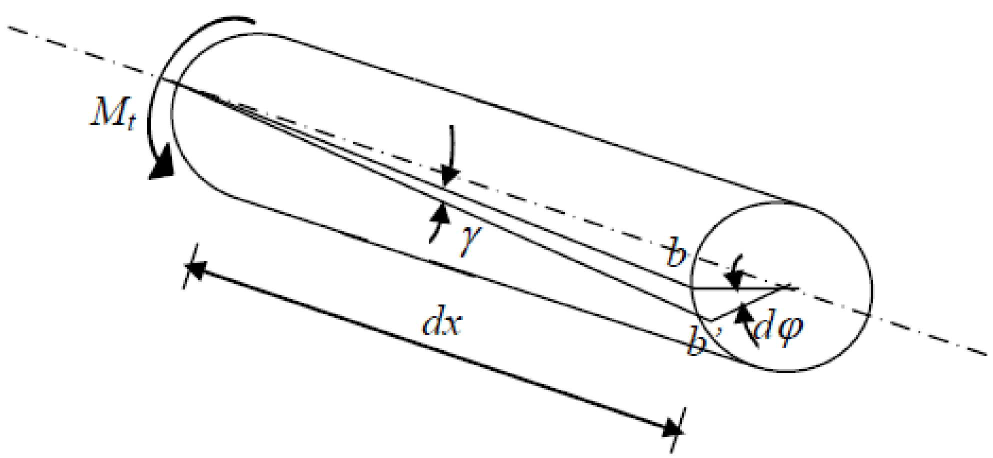


Fig 5.3. Element of a bar subjected to torsion

$$bb' = rd\varphi = \gamma dx$$

$$\Longrightarrow \gamma = r\frac{d\varphi}{dx}$$

γ is called the shear deformation (or shear angle).

$\frac{d\varphi}{dx}$ is the angle of torsion per unit length, which is constant and denoted by θ, such that:

$$\gamma = r\theta \qquad 5.2$$

5.3.3. Shear stresses

Hooke's law for shear is:

$$\tau = \gamma . G \qquad 5.3$$

G is called the shear modulus; it depends on the material and is expressed in MPa.

By substituting the expression of the shear strain, we obtain:

$$\tau = r.\theta.G \qquad 5.4$$

To calculate the shear stresses in the bar, instead of using the radius r, we use the polar coordinate ρ at the point considered:

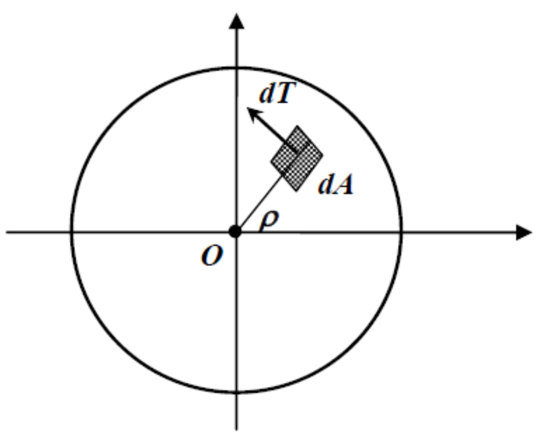


Fig 5.4. Shear force at a point on a cross-section

The elemental force acting on the surface dA is:

$$dT = \tau dA$$

This force produces an elemental torsional moment dM_t:

$$dM_t = \rho dT$$

Thus, the total torsional moment is:

$$M_t = \int \rho.\tau.dA = G\theta \int \rho^2 dA \qquad 5.5$$

The quantity $\int \rho^2 dA$ is called the polar moment of inertia **(**I_P). Hence,

$$M_t = G.\theta.I_P$$

$$\theta = \frac{M_t}{G.I_P} \qquad 5.6$$

From relations (5.2), (5.3), and (5.6), we get:

$$\tau = \frac{M_t}{I_P}\rho \qquad 5.7$$

The maximum shear stress is:

$$\tau_{max} = \frac{M_t}{I_P}r = \frac{M_t}{W_P} \qquad 5.8$$

Where $W_P = \frac{I_p}{r}$ is the polar section modulus

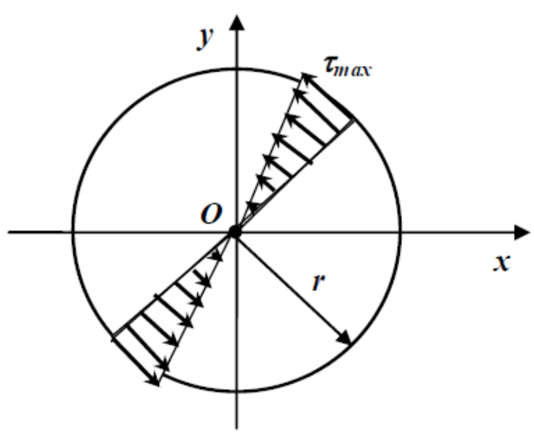
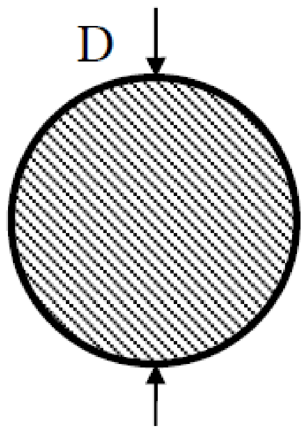


Fig 5.5. Distribution of shear stresses in a cross-section subjected to torsion

Examples of calculating polar section moduli

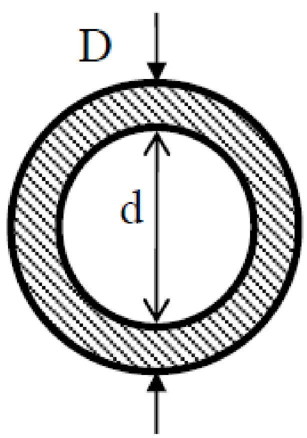


Solid circular section:

$$I_p = \frac{\pi D^2}{32}$$

$$r = \frac{D}{2}$$

$$W_p = \frac{\pi D^3}{16}$$



Tubular circular section

$$I_p = \frac{\pi (D^4 - d^4)}{32}$$

$$r = \frac{D}{2}$$

$$W_p = \frac{\pi (D^4 - d^4)}{16D}$$

5.4. Torsional Design

5.4.1. Strength condition

For a body subjected to pure torsion to safely withstand the load, the maximum shear stresses τ_{max} produced by a given torsional moment in a section must be less than or equal to the allowable shear stress $[\tau]$ of the material forming the body, that is:

$$\tau_{max} \leq [\tau] \qquad 5.9$$

5.4.2. Rigidity condition

The following expression gives the relative angle of torsion: twisting

$$\theta = \frac{d\varphi}{dx} = \frac{M_t}{GI_p}$$

$$\Longrightarrow \varphi = \int_0^L \frac{M_t}{G.I_p}\, dx \qquad 5.10$$

For a portion of the bar of length L and constant cross-section (I_p= constant), subjected to a constant torsional moment (M_t=constant), the angle of torsion becomes:

$$\varphi = \frac{M_t}{G.I_p}\int_0^L dx = \frac{M_t.L}{G.I_p} \qquad 5.11$$

For the entire bar (composed of several segments):

$$\varphi = \sum_i \frac{M_{t_i}.L_i}{G.I_{p_i}} \qquad 5.12$$

The rigidity condition requires that the angle of torsion be less than or equal to the allowable value, that is:

$$\varphi \leq [\varphi] \qquad 5.13$$

Generally, $[\varphi]$ =0.3°/m

5.5. Torsion of a Bar with a Non-Circular Cross Section

The torsion of bars with non-circular cross sections cannot be studied using the methods of strength of materials. For this reason, some results obtained from the theory of elasticity are given here (without demonstration).

The maximum shear stresses in a section are given by:

$$\tau_{max} = \frac{M_t}{W_t} \qquad 5.14$$

Where W_t is called the torsional section modulus, and it differs from the polar section modulus W_t, except for circular sections. It is expressed by the relation:

$$W_t = \alpha.h.b^2 \qquad 5.15$$

Where h and b are the dimensions of the section, with h > b

The angle of torsion in this case is calculated using the expression:

$$\varphi = \frac{M_t.L}{G.I_t} \qquad 5.16$$

Where I_t is given by: $I_t = \eta.h.b^3$

The coefficients α and η are given as functions of the ratio h/b in Table 5.1.

Table 5.1. Values of the coefficients α and η as a function of the ratio h/b.

h/b	1.0	1.2	1.5	2.0	2.5	3.0	4.0	5.0	10.0	∞
α	0.208	0.219	0.231	0.246	0.258	0.267	0.282	0.291	0.313	0.333
η	0.141	0.166	0.196	0.229	0.249	0.263	0.281	0.291	0.313	0.333