Solution of Series of Exercises 06

Solution of Exercise 1

1. Since $2+\sin(xy+x)>0$, for all x and y, then the function:

$$F(y)=\int_0^{\frac{\pi}{2}}\frac{\cos(xy+x)}{2+\sin(xy+x)}dx,$$

is well defined on $\in\mathbb{R}$. This is therefore a proper integral.
Let $f(x,y)=\dfrac{\cos(xy+x))}{2+\sin(xy+x)}$. This function is continuous on $\left[0,\dfrac{\pi}{2}\right]\times\mathbb{R}$.
Therefore, the function F is continuous. We then have:

$$\begin{aligned}\lim_{y\to 0}\int_0^{\frac{\pi}{2}}\frac{\cos(xy+x)}{2+\sin(xy+x)}dx &= \int_0^{\frac{\pi}{2}}\lim_{y\to 0}\frac{\cos(xy+x)}{2+\sin(xy+x)}dx\\ &= \int_0^{\frac{\pi}{2}}\frac{\cos(x)}{2+\sin(x)}dx=\left[\ln(2+\sin(x))\right]_0^{\frac{\pi}{2}}\\ &= \ln\left(\frac{3}{2}\right).\end{aligned}$$

2. This is a generalized integral.
Let $f(x,y)=\dfrac{1}{x^2+y^2+1}$. This function is continuous on $[0,+\infty[\times\mathbb{R}$.
Furthermore, for all $(x,y)\in[0,+\infty[\times\mathbb{R}$, we have:

$$\frac{1}{x^2+y^2+1}\leq\frac{1}{x^2+1}.$$

Since the integral $\int_0^{+\infty}\dfrac{dx}{x^2+1}$ converges, the Weierstrass criterion ensures the uniform convergence of the integral. $\int_0^{+\infty}\dfrac{dx}{x^2+y^2+1}$ on $\mathbb{R}$. Therefore, the function F defined on $\mathbb{R}$ by:

$$F(y)=\int_0^{+\infty}\frac{dx}{x^2+y^2+1},$$

is continuous.

Solution of Exercise 1

We then have:

$$\begin{aligned}
\lim_{y\to 0}\int_0^{+\infty}\frac{dx}{x^2+y^2+1} &= \int_0^{+\infty}\lim_{y\to 0}\frac{dx}{x^2+y^2+1}\\
&= \int_0^{+\infty}\frac{dx}{x^2+1} = \lim_{t\to+\infty}\left[\arctan(x)\right]_0^t\\
&= \frac{\pi}{2}.
\end{aligned}$$

Solution of Exercise 2

1.Let us show that the integral $I(y)=\int_0^{+\infty} y\exp(-xy)dx$ converges simply on $[0,1]$.

Let $f(x,y)=y\exp(-xy)$.

* If $y=0$, $f(x,0)=0$, which implies that $I(0)=\int_0^{+\infty} 0dx=0$.

* If $y\in\,]0,1]$, we have $I(y)=\int_0^{+\infty} y\exp(-xy)dx=-\lim_{t\to+\infty}\left[\exp(-xy)\right]_0^t=1$.

Consequently $I(y)=\int_0^{+\infty} y\exp(-xy)dx$ converge simplement sur $[0,1]$, and moreover:

$$I(y)=\begin{cases}0, \text{ if } y=0\\ 1, \text{ if } y\in\,]0,1]\,.\end{cases}$$

Since the function is not continuous, we deduce that the integral in question does not converge uniformly on $[0,1]$.

2. Let's show that $I(y)=\int_0^{+\infty} y\exp(-xy)dx$ converges uniformly on all segment $]0,1]$.

Let $\delta>0$. For all $y\in[\delta,1]$, we have:

$$y\exp(-xy)\le\exp(-x\delta).$$

Since the integral $\int_0^{+\infty}\exp(-x\delta)dx=1$ converges, The Weierstrass criterion ensures the uniform convergence of the integral $\int_0^{+\infty} y\exp(-xy)$ on $[\delta,1]$, for all $\delta>0$, therefore it converges uniformly on $]0,1]$.

Solution of Exercise 3

1. Let's study if the generalized integral

$$I(\alpha,\beta)=\int_0^{+\infty}\exp(-\alpha x)\ \frac{\sin(\beta x)}{x}dx,\ \alpha\geq 0.$$

does it make sense or not and let's give them their possible value.

Let α be a fixed, strictly positive real number.

Let:

$$f(x,\beta)=\begin{cases}\exp(-\alpha x)\ \dfrac{\sin(\beta x)}{x},\ \text{if } x\neq 0\\ \beta,\ \text{if } x=0\end{cases}$$

It suffices to demonstrate the following properties:

1. The functions f and its derivative with respect to β are continuous on $[0,+\infty[\times\mathbb{R}$. 2. The integral $\int_0^{+\infty} f(x,\beta)dx$ converges simply on $\mathbb{R}$

3. The integral $\int_0^{+\infty}\frac{\partial f}{\partial\beta}(x,\beta)dx$ converges uniformly on $\mathbb{R}$.

For all $x\in[0,+\infty[$ fixed, f is derivable with respect to β on $\mathbb{R}$, and we have:

$$\frac{\partial f}{\partial\beta}(x,\beta)=\exp(-\alpha x)\cos(\beta x).$$

It is clear that f and $\frac{\partial f}{\partial\beta}$ are continuous on $[0,+\infty[\times\mathbb{R}$.

Furthermore, the function $x\mapsto\dfrac{\exp(-\alpha x)}{x}$ is decreasing and tends towards 0, when $x\longrightarrow+\infty$, and $\left|\int_0^t\sin(\beta x)dx\right|\leq 2$ for all $t\in[0,+\infty[$.

So according to Abel, the integral $\int_0^{+\infty} f(x,\beta)dx$ converges simply on $\mathbb{R}$.

On the other hand, for all $\alpha>0$ and for all $\beta\in\mathbb{R}$, we have:

$$\left|\frac{\partial f}{\partial\beta}(x,\beta)\right|=\left|\exp(-\alpha x)\cos(\beta x)\right|\leq\left|\exp(-\alpha x)\right|.$$

Since $\int_0^{+\infty}\exp(-\alpha x)dx$ converges, the Weierstrass criterion affirms the uniform convergence of the integral $\int_0^{+\infty}\frac{\partial f}{\partial\beta}(x,\beta)dx$.

Solution of Exercise 3

The properties 1, 2 and 1 are therefore verified, which shows the derivability of the integral $\int_0^{+\infty} f(x,\beta)dx$ with respect to β on $\mathbb{R}$ and furthermore:

$$\begin{aligned}
\frac{\partial I}{\partial \beta}(\alpha,\beta) &= \frac{\partial}{\partial \beta}\int_0^{+\infty} f(x,\beta)dx \\
&= \int_0^{+\infty} \frac{\partial f}{\partial \beta}(x,\beta)dx \\
&= \int_0^{+\infty} \exp(-\alpha x)\cos(\beta x)dx \\
&= \Re(\int_0^{+\infty} \exp\left((-\alpha + i\beta)\,x\right)dx) \\
&= \frac{\alpha}{\alpha^2+\beta^2}
\end{aligned}$$

We perform an integration with respect to β, we get:

$$I(\alpha,\beta) = \arctan\left(\frac{\beta}{\alpha}\right) + C(\alpha),$$

where C is a function of a single variable α.

We then have:

$$\arctan\left(\frac{0}{\alpha}\right) + C(\alpha) = I(\alpha,0) = \int_0^{+\infty} 0dx = 0$$

So$C(\alpha) = 0$.

Finally:

$$\begin{aligned}
I(\alpha,\beta) &= \int_0^{+\infty} \exp(-\alpha x)\ \frac{\sin(\beta x)}{x}dx \\
&= \arctan\left(\frac{\beta}{\alpha}\right).
\end{aligned}$$

2. Let's find the value of the Dirichlet integral $D(\beta) = \int_0^{+\infty} \frac{\sin(\beta x)}{x}dx.$

Let β be fixed and consider the integral:

$$I(\alpha) = \int_0^{+\infty} \frac{\exp(-\alpha x)}{x}\sin(\beta x)dx.$$

According to Abel, this last integral is uniformly convergent on$[0,+\infty[$.

Solution of Exercise 3

Indeed, for all $\alpha \geq 0$ and for all $x \geq 0$, we have:

$$\frac{\exp(-\alpha x)}{x} \leq \frac{1}{x},$$

quantity that tends towards 0 uniformly when $x \to +\infty$. Furthermore

$$\left|\int_0^t \sin(\beta x) dx\right| \leq 2, \text{ for all } t \in [0, +\infty[\,.$$

According to Abel, the integral $I(\alpha)$ converges uniformly on $[0, +\infty[$.

So $I(\alpha)$ is continuous on $[0, +\infty[$ and we have:

$$\begin{aligned} D(\beta) &= \int_0^{+\infty} \frac{\sin(\beta x)}{x} dx \\ &= \lim_{\alpha \to 0^+} I(\alpha) \\ &= \lim_{\alpha \to 0^+} \arctan\left(\frac{\beta}{\alpha}\right), \text{ for } \beta \neq 0 \\ &= \frac{\pi}{2} sgn\,(\beta)\,, \text{ for} \beta \neq 0. \end{aligned}$$

If $\beta = 0$, we have $D(\beta) = 0$.

3. We can therefore deduce that:

1.3

$$\begin{aligned} I_1 &= \int_0^{+\infty} \frac{\sin(\alpha x)\cos(\beta x)}{x} dx \\ &= \frac{1}{2}\int_0^{+\infty} \left(\frac{\sin\left((\alpha+\beta)\,x\right)}{x} + \frac{\sin\left((\alpha-\beta)\,x\right)}{x}\right) dx \\ &= \frac{1}{2}\left(\int_0^{+\infty} \frac{\sin\left((\alpha+\beta)\,x\right)}{x} dx + \int_0^{+\infty} \frac{\sin\left((\alpha-\beta)\,x\right)}{x} dx\right) \\ &= \frac{\pi}{4}\left(sgn\,(\alpha+\beta) + sgn\,(\alpha-\beta)\right). \end{aligned}$$

2.3. By changing the variable $x^2 = y$, we get:

$$I_2 = \int_0^{+\infty} \frac{\sin(x^2)}{x} dx = \frac{1}{2}\int_0^{+\infty} \frac{\sin(y)}{y} dy = \frac{\pi}{4}.$$

Solution of Exercise 3

3.3. We have:

$$\begin{aligned}
I_3 &= \int_0^{+\infty} \frac{\sin^3(\alpha x)}{x} dx \\
&= \int_0^{+\infty} \left(\frac{3}{4} \frac{\sin(\alpha x)}{x} - \frac{1}{4} \frac{\sin(3\alpha x)}{x} \right) dx \\
&= \frac{3}{4} \int_0^{+\infty} \frac{\sin(\alpha x)}{x} dx - \frac{1}{4} \int_0^{+\infty} \frac{\sin(3\alpha x)}{x} dx \\
&= \frac{3\pi}{8} sgn(\alpha) - \frac{\pi}{8} sgn(3\alpha) \\
&= \frac{3\pi}{8} sgn(\alpha) - \frac{\pi}{8} sgn(\alpha) \\
&= \frac{\pi}{4} sgn(\alpha).
\end{aligned}$$

Solution of Exercise 4

Establish the following relationship:

$$F(\alpha) = \int_0^{\frac{\pi}{2}} \ln\left(\alpha^2 \sin^2 x + \beta^2 \cos^2\right) dx = \pi \ln\left(\frac{|\alpha| + |\beta|}{2}\right).$$

Let $f(x, \alpha, \beta) = \ln\left(\alpha^2 \sin^2 x + \beta^2 \cos^2\right)$. For all $(x, \alpha, \beta) \in \left[0, \frac{\pi}{2}\right] \times \mathbb{R}_*^2$, we have:

$$\alpha^2 \sin^2 x + \beta^2 \cos^2 > 0.$$

This is therefore a proper integral.

The functions $f(x, \alpha, \beta)$ and $\frac{\partial f}{\partial \alpha}(x, \alpha, \beta) = \frac{2\alpha \sin^2 x}{\alpha^2 \sin^2 x + \beta^2 \cos^2}$ are continuous on $\left[0, \frac{\pi}{2}\right] \times \mathbb{R}$.

We then have:

$$\begin{aligned}
\dot{F}(\alpha) &= \frac{\partial}{\partial \alpha} \int_0^{\frac{\pi}{2}} f(x, \alpha, \beta) dx = \int_0^{\frac{\pi}{2}} \frac{\partial f}{\partial \alpha}(x, \alpha, \beta) dx \\
&= \int_0^{\frac{\pi}{2}} \frac{2\alpha \sin^2 x}{\alpha^2 \sin^2 x + \beta^2 \cos^2} dx = \int_0^{\frac{\pi}{2}} \frac{2\alpha}{\alpha^2 + \beta^2 \left(\frac{\cos x}{\sin x}\right)^2} dx, \ x \neq 0
\end{aligned}$$

Solution of Exercise 4

By performing the variable change $y = \dfrac{\cos x}{\sin x}$, we get:

$$\begin{cases} dx = \dfrac{-dy}{1+y^2} \\ y \in [0, +\infty[\end{cases}$$

So:

$$\begin{aligned}
\dot{F}(a) &= \int_0^{+\infty} \frac{2\alpha}{(\alpha^2+\beta^2y^2)(1+y^2)}dy \\
&= \frac{2\alpha}{\alpha^2-\beta^2}\int_0^{+\infty} \left(\frac{-\beta^2}{\alpha^2+\beta^2y^2}+\frac{1}{1+y^2}\right)dy \\
&= \frac{2\alpha}{\alpha^2-\beta^2}\int_0^{+\infty} \left(\frac{-\beta^2}{\alpha^2\left(1+\left(\left|\frac{\beta}{\alpha}\right|y\right)^2\right)}+\frac{1}{1+y^2}\right)dy \\
&= \frac{2\alpha}{\alpha^2-\beta^2}\left(-\frac{\beta^2}{\alpha^2}\left|\frac{\alpha}{\beta}\right|\lim_{y\to+\infty}\arctan y + \lim_{y\to+\infty}\arctan y\right) \\
&= \frac{\alpha\pi}{\alpha^2-\beta^2}\left(\frac{\beta^2}{\alpha^2}\left|\frac{\alpha}{\beta}\right|+1\right) \\
&= \frac{\alpha\pi}{\alpha^2-\beta^2}\left(-\left|\frac{\beta}{\alpha}\right|+1\right) \\
&= \frac{\alpha\pi}{\left(|\alpha|-|\beta|\right)\left(|\alpha|+|\beta|\right)}\left(\frac{|\alpha|-|\beta|}{|\alpha|}\right) \\
&= \frac{\alpha\pi}{|\alpha|\left(|\alpha|+|\beta|\right)} \\
&= \frac{\pi sgn(\alpha)}{\left(|\alpha|+|\beta|\right)}.
\end{aligned}$$

By taking a simple integral, we find:

$$F(\alpha) = \pi\ln\left(|\alpha|+|\beta|\right) + C(\alpha).$$

Let's calculate $C(\alpha)$. We have:

$$\begin{aligned}
F(\beta) &= \pi\ln\left(2|\beta|\right) + C(\alpha) \\
&= \pi\ln(2) + \pi\ln\left(|\beta|\right) + C(\alpha). \qquad (1)
\end{aligned}$$

Solution of Exercise 4

On the other hand:

$$\begin{aligned} F(\beta) &= \int_0^{\frac{\pi}{2}} \ln\left(\beta^2 \sin^2 x + \beta^2 \cos^2\right) dx \\ &= 2\int_0^{\frac{\pi}{2}} \ln\left(\left|\beta\right|\right) dx = \pi \ln\left(\left|\beta\right|\right) \end{aligned}$$

From (1) and (2), we get $C(\alpha) = -\pi \ln(2)$.

Finally:

$$\begin{aligned} F(\alpha) &= \int_0^{\frac{\pi}{2}} \ln\left(\alpha^2 \sin^2 x + \beta^2 \cos^2\right) dx \\ &= \pi \ln\left(|\alpha| + \left|\beta\right|\right) - \pi \ln(2) \\ &= \pi \ln\left(\frac{|\alpha| + \left|\beta\right|}{2}\right). \end{aligned}$$

Solution of Exercise 5

1. Let $y = \frac{x}{2}$. So:

$$\begin{aligned} I_1 &= \int_0^2 \frac{dx}{\sqrt{2x - x^2}} = \int_0^1 \frac{dy}{\sqrt{y - (1-y)}} \\ &= \int_0^1 y^{\frac{-1}{2}} (1-y)^{\frac{-1}{2}}\, dy = \int_0^1 y^{\frac{1}{2}-1} (1-y)^{\frac{1}{2}-1}\, dy \\ &= B\left(\frac{1}{2}, \frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) \times \Gamma\left(\frac{1}{2}\right) = \Gamma\left(1 - \frac{1}{2}\right) \times \Gamma\left(\frac{1}{2}\right) \\ &= B\left(1 - \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi. \end{aligned}$$

Solution of Exercise 5

2. Let $y = \ln x$. So:

$$\begin{aligned}
I_2 &= \int_1^{+\infty} \frac{\ln(\ln(x))\,dx}{x\sqrt{\ln x}\,(1+\ln x)} = \int_0^{+\infty} \frac{\ln(y)\,dy}{y^{\frac{1}{2}}(1+y)} \\
&= \int_0^{+\infty} \frac{y^{\frac{1}{2}-1}\ln(y)\,dy}{(1+y)} = \frac{\partial}{\partial\alpha} B(\alpha, 1-\alpha)_{\alpha=\frac{1}{2}} \\
&= \frac{\partial}{\partial\alpha}\left(\frac{\pi}{\sin\alpha\pi}\right)_{\alpha=\frac{1}{2}} = 0.
\end{aligned}$$

3. Let $\exp x = 1 + y$. So:

$$\begin{aligned}
I_3 &= \int_0^{+\infty} \exp(-3x)(\exp(x)-1)^{\frac{3}{2}}\,dx = \int_0^{+\infty} y^{\frac{3}{2}}(1+y)^{-4}\,dy \\
&= \int_0^{+\infty} \frac{y^{\frac{3}{2}}}{(1+y)^4}dy = \int_0^{+\infty} \frac{y^{\frac{5}{2}-1}}{(1+y)^{\frac{5}{2}+\frac{3}{2}}}dy \\
&= B\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\times\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} \\
&= \frac{\Gamma\left(\frac{3}{2}+1\right)\times\Gamma\left(\frac{3}{2}\right)}{6} = \frac{3}{2}\times\frac{\Gamma\left(\frac{3}{2}\right)\times\Gamma\left(\frac{3}{2}\right)}{6} \\
&= \frac{1}{4}\times\Gamma\left(\frac{1}{2}+1\right)\times\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{16}\times\Gamma\left(\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{16}\times\Gamma\left(1-\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right) = \frac{1}{16}\times B\left(1-\frac{1}{2}, \frac{1}{2}\right) \\
&= \frac{1}{16}\times\frac{\pi}{\sin(\frac{\pi}{2})} = \frac{\pi}{16}
\end{aligned}$$

4. We have:

$$\begin{aligned}
I_4 &= \int_0^{+\infty} \frac{t^{\frac{5}{3}-1}}{(1+t)^{\frac{5}{3}+\frac{4}{3}}}dt = B\left(\frac{5}{3}, \frac{4}{3}\right) \\
&= \frac{\Gamma(\frac{5}{3})\Gamma(\frac{4}{3})}{\Gamma(3)} = \frac{\Gamma(\frac{2}{3}+1)\Gamma(\frac{1}{3}+1)}{2} \\
&= \frac{\frac{2}{3}\times\Gamma(\frac{2}{3})\times\frac{1}{3}\Gamma(\frac{1}{3})}{2} = \frac{1}{9}\frac{\Gamma(1-\frac{1}{3})\times\Gamma(\frac{1}{3})}{\Gamma(1-\frac{1}{3}+\frac{1}{3})} \\
&= B\left(\frac{1}{3}, 1-\frac{1}{3}\right) = \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}.
\end{aligned}$$