Series of Exercises 06

Exercise 1

Study whether the following limits make sense or not and give their possible values:

❶ $\lim\limits_{y\to 0}\int\limits_{-1}^{1}\dfrac{dx}{x^2+y^2+1}$.

❷ $\lim\limits_{y\to 0}\int\limits_{0}^{\frac{\pi}{2}}\dfrac{\cos(xy+x)}{2+\sin(xy+x)}dx$.

Exercise 2

❶ Show that the generalized integral:

$$I(y)=\int_0^{+\infty} y\exp(-xy)dx,$$

converges simply on $[0,1]$, but it does not converge uniformly on the same segment.

❷ Show that this integral converges uniformly on every segment $]0,1]$.

Exercise 3

❶ Study whether the following integral makes sense or not and give their possible value:

$$I(\alpha,\beta)=\int_0^{+\infty}\exp(-\alpha x)\ \frac{\sin(\beta x)}{x}dx,\ \alpha\geq 0.$$

❷ By taking the limit with appropriate justification, find the value of the Dirichlet integral:

$$D(\beta)=\int_0^{+\infty}\ \frac{\sin(\beta x)}{x}dx.$$

❸ Deduce the values of the following integrals:

$$I_1=\int_0^{+\infty}\ \frac{\sin(\alpha x)\cos(\beta x)}{x}dx,\quad I_2=\int_0^{+\infty}\ \frac{\sin(x^2)}{x}dx,\quad I_3=\int_0^{+\infty}\ \frac{\sin^3(\alpha x)}{x}dx.$$

Exercise 4

Let α and β be two non-zero real numbers. By performing a differentiation with respect to the parameter α, establish the following relation:

$$F(a) = \int_0^{\frac{\pi}{2}} \ln\left(\alpha^2 \sin^2 x + \beta^2 \cos^2\right) dx = \pi \ln\left(\frac{|\alpha| + |\beta|}{2}\right)$$

Exercise 5

Using the special functions, calculate the following integrals:

❶ $I_1 = \int_0^{\frac{1}{2}} x^2 \sqrt{1 - 4x^2} dx$ (by setting $x = \frac{y}{2}$).

❷ $I_2 = \int_1^{+\infty} \frac{\ln(\ln(x))\, dx}{x \sqrt{\ln x}\,(1 + \ln x)}$ (by setting $y = \ln x$).

❸ $I_3 = \int_0^{+\infty} \exp(-3x)\,(\exp(x) - 1)^{\frac{3}{2}}\, dx$ (by setting $\exp x = 1 + y$).

❹ $I_4 = \int_0^{+\infty} \frac{x^{\frac{2}{3}}}{(1 + x)^3} dx.$