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Functions Defined by Integrals

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Chapter 1

Functions defined by an integral

This chapter consists of studying functions of the form:

$$F(y) = \int_a^b f(x, y)dx, \text{ where } a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \cup \{+\infty\},$$

in particular, we ask ourselves the question of knowing on what conditions on f we have the continuity, the differentiability and the integrability of the function F. We distinguish two cases: functions defined by proper integrals or functions defined by improper integrals.

1.1 Functions defined by a proper integral

In this section we consider a function $f = f(x, y) : [a, b] \times I \to \mathbb{R}$ (I an open interval of $\mathbb{R}$) Riemann-integrable with respect to the first variable x on $[a, b]$, for all $y \in I$.

1.1.1 Properties of a function defined by a proper integral

Continuity

In this paragraph, we are interested in the continuity of the function defined by:

$$F(y) = \int_a^b f(x, y)dx, y \in I. \tag{1.1}$$

Theorem 1.1.1. *Let $f : [a, b] \times I \longrightarrow \mathbb{R}$ be a continuous function, then the function F defined by the relation (1.1) is also continuous on I. In particular, for all $y_0 \in I$, we have:*

$$\begin{aligned} \lim_{y \to y_0} F(y) &= \lim_{y \to y_0} \int_a^b f(x, y)dx = \int_a^b \lim_{y \to y_0} f(x, y)dx \\ &= \int_a^b f(x, y_0)dx = F(y_0), \end{aligned}$$

which is a case of ***interversion of limit and integral****.*

Proof. Since f is continuous, the function $x \mapsto f(x, y)$ is integrable on $[a, b]$. Let now y_1, y_2 be any two points of I and let $V \subset I$ be a compact interval containing y_1 and y_2.
Since f is continuous on $[a, b] \times I$, it is therefore uniformly continuous on the compact $[a, b] \times V$, i.e;
$\forall \epsilon > 0, \exists \delta > 0$, such that $\forall (x_1, y_1), (x_2, y_2) \in [a, b] \times V$, we have:

$$\left\| (x_1, y_1) - (x_2, y_2) \right\| < \delta \Rightarrow \left| f(x_1, y_1) - f(x_2, y_2) \right| < \frac{\epsilon}{b - a}. \tag{1.2}$$

In particular, if we set $x = x_1 = x_2$, we find, for all $y_1, y_2 \in V$:

$$\left| y_1 - y_2 \right| < \delta \Rightarrow \left| f(x, y_1) - f(x, y_2) \right| < \frac{\epsilon}{b - a}. \tag{1.3}$$

Hence, for all $y_1, y_2 \in V$, such that $\left| y_1 - y_2 \right| < \delta$, we have:

$$\begin{aligned} \left| F(y_1) - F(y_2) \right| &= \left| \int_a^b (f(x, y_1) - f(x, y_2)) \, dx \right| \\ &\leq \int_a^b \left| f(x, y_1) - f(x, y_2) \right| dx < \epsilon. \end{aligned} \tag{1.4}$$

Consequently F is uniformly continuous on V, so it is continuous on I. ◻

Derivability

Theorem 1.1.2. *Let $f : [a,b] \times I \to \mathbb{R}$ be a continuous function on $[a,b] \times I$. We assume that the partial derivative $\dfrac{\partial f}{\partial y}$ exists and is continuous on $[a,b] \times I$, then the function F defined by the relation (1.1) is derivable on I, and we have:*

$$\dot{F}(y) = \frac{\partial}{\partial y}\left(\int_a^b f(x,y)dx\right) = \int_a^b \frac{\partial}{\partial y} f(x,y)dx, \tag{1.5}$$

which is a case of ***inversion of derivative and integral****.*

Proof. Let us first note that the functions $x \mapsto f(x,y)$ and $x \mapsto \dfrac{\partial}{\partial y} f(x,y)$ are integrable on $[a,b]$ because f and $\dfrac{\partial}{\partial y} f$ are continuous.
As before, let y be any point of I and let $V \subset I$ be a compact interval containing y. It suffices to show that:

$$\lim_{h\to 0}\left(\frac{F(y+h)-F(y)}{h}\right) - \int_a^b \frac{\partial}{\partial y} f(x,y)dx = 0. \tag{1.6}$$

Indeed, if the Mean Value Theorem (MVT) applies, we have then:

$$\begin{aligned} \frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial}{\partial y} f(x,y)dx &= \int_a^b \left(\frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial}{\partial y} f(x,y)\right)dx \quad (1.7)\\ &= \int_a^b \left(\frac{\partial}{\partial y} f(x,y+\theta h) - \frac{\partial}{\partial y} f(x,y)\right)dx,\ \theta \in \left]0,1\right[. \end{aligned}$$

Since $\dfrac{\partial}{\partial y} f$ is uniformly continuous on the compact $[a,b] \times V$, we then have, $\forall \epsilon > 0, \exists \delta > 0$, such that $\forall x \in [a,b]\,, \forall y, h \in V$:

$$\left|(y+h)-y\right| = |h| < \delta \Rightarrow \left|\frac{\partial}{\partial y} f(x,y+\theta h) - \frac{\partial}{\partial y} f(x,y)\right| < \frac{\epsilon}{b-a}. \tag{1.8}$$

Consequently:

$$\left|\frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial}{\partial y} f(x,y)dx\right| \le \int_a^b \left|\frac{\partial}{\partial y} f(x,y+\theta h) - \frac{\partial}{\partial y} f(x,y)\right| < \epsilon, \tag{1.9}$$

which proves the derivability of F at y, and since y is considered arbitrary in I, this clearly shows the derivability of F on I. □

Integration

Theorem 1.1.3. *Let $f : [a, b] \times I \to \mathbb{R}$ be a continuous function on $[a, b] \times I$, then the function F defined by the relation (1.1) is integrable on I. In particular for all $y \in I$, we have:*

$$\begin{aligned} \int_I F(y)dy &= \int_I dy\left(\int_a^b f(x,y)dx\right) \\ &= \int_a^b dx\left(\int_I f(x,y)dy\right), \end{aligned} \tag{1.10}$$

which is a case of ***interversion of integrals****.*

Proof. We are going to demonstrate a more general formula. That is, we want to demonstrate that:

$$G(z) = \int_I dy\left(\int_a^z f(x,y)dx\right) = \int_a^z dx\left(\int_I f(x,y)dy\right) = H(z), \text{ for allt } z \in [a,b]\,, \tag{1.11}$$

where G, H are continuous functions on $[a, b]$.
Let $\int_a^z f(x, y)dx = F(z, y)$. It is clear that F is continuous by the theorem 1.1.1, and moreover:

$$\frac{\partial}{\partial z}F(z,y) = f(z,y). \tag{1.12}$$

We deduce from the theorem 1.1.2 that:

$$\dot{G}(z) = \int_I \frac{\partial}{\partial z}F(z,y)dy = \int_I f(z,y)dy. \tag{1.13}$$

On the other hand, if we set $\int_c^d f(x, y)dy = K(x)$, we can then write:

$$H(z) = \int_a^z K(x)dx. \tag{1.14}$$

Since k is continuous, we deduce:

$$\dot{H}(z) = K(z) = \int_c^d f(z,y)dy \tag{1.15}$$

From (1.13) and (1.15), we find:

$$\dot{G}(z) = \dot{H}(z) \tag{1.16}$$

By integrating this last equality from a to t, and using the fact that $G(a) = H(a) = 0$, we deduce that $G(z) = H(z)$, for all $z \in [a, b]$. □

1.2 Functions defined by a generalized integral

Let $f : [a, b[\times I \to \mathbb{R}$ (I an open interval of $\mathbb{R}$) be a function of two variables, $b \in \mathbb{R} \cup \{+\infty\}$ and that for some $y \in I$, the function f is not defined at b. We also assume that the generalized integral $\int_a^b f(x, y)dx$ converges and we are interested in the properties of the functions defined on I by the following generalized integrals:

$$F(y) = \int_a^{+\infty} f(x, y)dx = \lim_{z\to+\infty} \int_a^{z} f(x, y)dx \tag{1.17}$$

and:

$$F^*(y) = \int_a^{b} f(x, y)dx = \lim_{\gamma\to 0} \int_a^{b-\gamma} f(x, y)dx. \tag{1.18}$$

1.2.1 Uniform convergence of generalized integrals

Definition 1.2.1. *We say that the generalized integral (1.17) is uniformly convergent on I, if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z \in [\delta, +\infty[,\ \left|F(y) - \int_a^{z} f(x, y)dx\right| < \epsilon. \tag{1.19}$$

Definition 1.2.2. *We say that the generalized integral (1.18) is uniformly convergent on I, if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall \gamma \in]0, \delta[\,,\ \left|F^*(y) - \int_a^{b-\gamma} f(x, y)dx\right| < \epsilon. \tag{1.20}$$

Theorem 1.2.1. *Let $f : [a, +\infty[\times I \to \mathbb{R}$ (I be an open interval of $\mathbb{R}$) an integrable function (in the generalized sense) on the interval $[a, +\infty[$. Then the following properties are equivalent:*

1. The integral $\int_a^{+\infty} f(x, y)dx$ converges uniformly on I.

2. The sequence of functions with general term $F_n(y) = \int\limits_a^{y_n} f(x, y)dx$ converges uniformly on I, where $(y_n)_n$ is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$.

Proof. The proof of this theorem follows immediately from the proof of Theorem 1.6.1 (See Chapter 5). □

Corollary 1.2.1. *Similarly, if f is an integrable function (in the generalized sense on the interval $[a, b[$. Then the following properties are equivalent:*
1. The integral $\int_a^b f(x, y)dx$ converges uniformly on I.
2. The sequence of functions with general term $F_n^*(y) = \int\limits_a^{\gamma_n} f(x, y)dx$ converges uniformly on I , where $(\gamma_n)_n$ is a sequence of elements of $[a, +b[$ with limit 0, when $n \to +\infty$.

1.2.2 Uniform convergence criteria

Cauchy criterion

Theorem 1.2.2. *A necessary and sufficient condition for the generalized integral (1.17) to be uniformly convergent on I is:*

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z_2 > z_1 \geq \delta,\ \left| \int_{z_1}^{z_2} f(x, y)dx \right| < \epsilon. \tag{1.21}$$

Proof. * Suppose that the generalized integral (1.17) is uniformly convergent on $[a, +\infty[$. We then have:

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z_2 > z_1 \geq \delta,\ \left| \int_{z_1}^{+\infty} f(x, y)dx \right| < \frac{\epsilon}{2} \text{ and } \left| \int_{z_2}^{+\infty} f(x, y)dx \right| < \frac{\epsilon}{2}.$$

For all $y \in I$, and for all $z_2 > z_1 \geq \delta$, we can write:

$$\begin{aligned} \left| \int_{z_1}^{z_2} f(x, y)dx \right| &= \left| \int_{z_1}^{+\infty} f(x, y)dx - \int_{z_2}^{+\infty} f(x, y)dx \right| \\ &\leq \left| \int_{z_1}^{+\infty} f(x, y)dx \right| + \left| \int_{z_2}^{+\infty} f(x, y)dx \right| < \epsilon. \end{aligned} \tag{1.22}$$

* Now, suppose that the generalized integral (1.17) verifies the Cauchy criterion, i.e:

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z_2 > z_1 \geq \delta, \left| \int_{z_1}^{z_2} f(x, y)dx \right| < \epsilon.$$

By passing to the limit, when $z_2 \to +\infty$, we obtain the requested result. □

Corollary 1.2.2. *In the same way, we can demonstrate that the generalized integral (1.18) is uniformly convergent on I, if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0, \ \forall y \in I, \ \forall \gamma_1, \gamma_2 \in]0, \delta[\,, (\gamma_1 > \gamma_2), \left| \int_{b-\gamma_1}^{b-\gamma_2} f(x,y)dx \right| < \epsilon. \tag{1.23}$$

Weierstrass criterion

Theorem 1.2.3. *Let $f : [a,b[\times I \to \mathbb{R}$ be an integrable function on $[a,b[$ (I an open interval of $\mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$). Suppose that there exists a real function g locally integrable on $[a,b[$ (called the upper bound function) verifying:*

$$\begin{cases} 1. \left|f(x,y)\right| \leq g(x), \text{ for all } x \in [a,b[, \\ 2. \int_a^b g(x)dx \text{ converges.} \end{cases}$$

Then, for all $y \in I$, the generalized integral $\int_a^b f(x,y)dx$ converges absolutely and uniformly on I.

Proof. The absolute convergence of the integral $\int_a^b f(x,y)dx$ follows immediately from the first condition.
The uniform convergence of the integral $\int_a^b f(x,y)dx$ follows immediately from the inequality:

$$\left| \int_{z_1}^{z_2} f(x,y)dx \right| \leq \int_{z_1}^{z_2} \left|f(x,y)\right| dx \leq \int_{z_1}^{z_2} \left|g(x)\right| dx < \epsilon,$$

and the Cauchy criterion. □

Abel-Dirichlet criterion

Lemma 1.2.1. *Let $f : [a,b[\times I \to \mathbb{R}$ be an integrable, positive and decreasing function with respect to x on $[a,b[$ and tending to 0 uniformly when x tends to b, and let $g : [a,b[\times I \to \mathbb{R}$ be an integrable function on $[a,b[$ verifying the following property:*

$$\exists M > 0, \forall z \in [a,b[\,, \forall y \in I, \left| \int_a^z g(x,y)dx \right| \leq M. \tag{1.24}$$

Then the integral $\int_a^b f(x,y)g(x,y)dx$ converges uniformly on I.

Proof. For all $z_1, z_2 \in [a, b[$, using the second formula of the mean value theoem of integrals, gives us:

$$\int_{z_1}^{z_2} f(x,y)g(x,y)dx = f(z_1,y)\int_{z_1}^{k} g(x,y)dx + f(z_2,y)\int_{k}^{z_2} g(x,y)dx,\ k \in [z_1, z_2]. \tag{1.25}$$

By hypothesis,

$$\lim_{x\to b} f(x,y) = 0 \Leftrightarrow \forall\epsilon > 0, \exists\delta > 0,\ \forall y \in I,\ \forall x \in [\delta, b[\,,\ \left|f(x,y)\right| < \frac{\epsilon}{4M}. \tag{1.26}$$

On the other hand, using the hypothesis (1.24), allows us to write:

$$\left|\int_{z_1}^{k} g(x,y)dx\right| = \left|\int_{a}^{k} g(x,y)dx - \int_{a}^{z_1} g(x,y)dx\right| \leq 2M,\ \text{for all } k \in [z_1, z_2]. \tag{1.27}$$

and

$$\left|\int_{k}^{z_2} g(x,y)dx\right| = \left|\int_{a}^{z_2} g(x,y)dx - \int_{a}^{k} g(x,y)dx\right| \leq 2M,\ \text{for all } k \in [z_1, z_2]. \tag{1.28}$$

So, for all $x, z_1, z_2 \in [\delta, b[$ and for all $k \in [z_1, z_2]$, we can write:

$$\begin{aligned}\left|\int_{z_1}^{z_2} f(x,y)g(x,y)dx\right| &\leq \left|f(z_1,y)\right|\left|\int_{z_1}^{k} g(x,y)dx\right| + \left|f(z_2,y)\right|\left|\int_{k}^{z_2} g(x,y)dx\right| \\ &< \frac{\epsilon}{4M}\times 2M + \frac{\epsilon}{4M} = \epsilon.\end{aligned}$$

□

1.2.3 Properties of a function defined by a generalized integral

In this section, we are interested in properties of the function defined for $y \in I$ by:

$$F(y) = \int_{a}^{+\infty} f(x,y)dx, \tag{1.29}$$

The case where b is fixed in $\mathbb{R}$ this deals with the same way.

Continuity

Theorem 1.2.4. *Let $f : [a, +\infty[\times I \to \mathbb{R}$ be a continuous function and let the integral (1.29) converge uniformly on I, then the function F defined by the relation (1.29) is also continuous on I. In particular, for all $y_0 \in I$, we have:*

$$\begin{aligned}\lim_{y\to y_0} F(y) &= \lim_{y\to y_0}\int_a^{+\infty} f(x,y)dx = \int_a^{+\infty}\lim_{y\to y_0} f(x,y)dx \\ &= \int_a^{+\infty} f(x,y_0)dx = F(y_0),\end{aligned} \tag{1.30}$$

which is a case of ***interversion of limit and integral.***

Proof. According to the theorem 1.2.1, the uniform convergence of the integral (1.29) implies the uniform convergence of the sequence of functions (F_n) with general term

$$F_n(y) = \int_a^{y_n} f(x,y)dx,$$

where (y_n) is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$. Using the theorem 1.1.1 shows that F_n is a continuous function on I, ; that is, $F(y) = \lim_{n\to+\infty}\int_a^{z_n} f(x,y)dx$ is a continuous function on I, and moreover:

$$\begin{aligned}\lim_{y\to y_0} F(y) &= \lim_{y\to y_0}\lim_{n\to+\infty}\int_a^{y_n} f(x,y)dx = \lim_{n\to+\infty}\lim_{y\to y_0}\int_a^{y_n} f(x,y)dx \\ &= \lim_{n\to+\infty}\int_a^{y_n}\lim_{y\to y_0} f(x,y)dx = \int_a^{+\infty} f(x,y_0)dx = F(y_0)\end{aligned}$$

□

Derivability

Theorem 1.2.5. *Let $f, \dfrac{\partial f}{\partial y} : [a, +\infty[\times I \to \mathbb{R}$ be two continuous functions. We assume that the integral (1.29) converges and that the integral $\int_a^{+\infty}\dfrac{\partial f}{\partial y}dx$ converges uniformly on I, then the function F defined by the relation (1.29) is derivable on I, and moreover:*

$$\dot{F}(y) = \frac{\partial}{\partial y}\left(\int_a^{+\infty} f(x,y)dx\right) = \int_a^{+\infty}\frac{\partial}{\partial y} f(x,y)dx, \tag{1.31}$$

which is a case ***of inversion of derivative and integral.***

Proof. According to the theorem 1.2.1, the convergence of the integral (1.29) leads to the convergence of the sequence of functions (F_n) of general term

$$F_n(y) = \int_a^{y_n} f(x,y)dx, \tag{1.32}$$

where (y_n) is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$. Since $f, \dfrac{\partial f}{\partial y} : [a, +\infty[\times I \to \mathbb{R}$ are continuous functions, using the theorem 1.1.2 shows that:

$$\dot{F}_n(y) = \int_a^{z_n} \frac{\partial f}{\partial y}(x,y)dx, \tag{1.33}$$

and moreover $\dot{F}_n$ is continuous on I (according to the theorem 1.1.1).
On the other hand, $\dot{F}_n$ is uniformly convergent on I because $\int_a^{+\infty} \frac{\partial}{\partial y} f(x,y)dx$ is also.
Now, using the theorem of the derivability of sequences of functions ensures the derivability of the function F on I, and moreover, we have:

$$\dot{F}(y) = \lim_{n\to+\infty} \dot{F}_n(y) = \lim_{n\to+\infty} \int_a^{z_n} \frac{\partial f}{\partial y}(x,y)dx = \int_a^{+\infty} \frac{\partial}{\partial y} f(x,y)dx.$$

□

Integration

Theorem 1.2.6. *Lete $f : [a, +\infty[\times I \to \mathbb{R}$ be a continuous function and let the integral (1.29) converge uniformly on I, then the function F defined by the relation (1.29) is also integrable on I, and moreover, we have:*

$$\int_I F(y)dy = \int_I dy\left(\int_a^{+\infty} f(x,y)dx\right) = \int_a^{+\infty} dx\left(\int_I f(x,y)dy\right). \tag{1.34}$$

which is a case of ***interversion of the two integrals.***

Proof. According to the theorem 1.2.1, the uniform convergence of the integral (1.29) implies the uniform convergence of the sequence of functions (F_n) with general term

$$F_n(y) = \int_a^{y_n} f(x,y)dx, \tag{1.35}$$

where (y_n) is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$. Using theorem 1.1.3 shows that F_n is an integrable function on I; that is,

$$F(y) = \lim_{n\to+\infty} \int_a^{z_n} f(x, y)dx$$

is an integrable function on I (limit of a continuous and uniformly convergent sequence of functions, so it is integrable), and moreover:

$$\begin{aligned}\int_I F(y)dy &= \int_I dy\left(\lim_{n\to+\infty}\int_a^{z_n} f(x,y)dx\right) = \lim_{n\to+\infty}\int_I dy\left(\int_a^{z_n} f(x,y)dx\right)\\ & \lim_{n\to+\infty}\int_a^{z_n} dx\left(\int_I f(x,y)dy\right).\end{aligned}$$

We then deduce:

$$\int_I F(y)dy = \int_I dy\left(\int_a^{+\infty} f(x,y)dx\right) = \int_a^{+\infty} dx\left(\int_I f(x,y)dy\right).$$

□

1.3 Special Functions

1.3.1 Euler's Gamma Function

Definition 1.3.1. *We call Euler's gamma function the special function Γ defined by:*

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}\exp(-x)dx,\ \alpha > 0. \tag{1.36}$$

Remak 1.1. *The Euler gamma function defined by the relation (1.36) is well defined on $]0, +\infty[$.*
Indeed: In the neighborhood of zero, $x^{\alpha-1}\exp(-x) \overset{V(0)}{\sim} x^{\alpha-1}$.
Since $\int_0^1 x^{\alpha-1}dx$ converges if and only if $\alpha > 0$, we deduce that $\int_0^1 x^{\alpha-1}\exp(-x)dx$ converges if and only if $\alpha > 0$.
In the neighborhood of infinity, if we set for example $g(x) = x^{-2}$, we then find:

$$\lim_{x\to+\infty}\frac{x^{\alpha-1}\exp(-x)}{g(x)} = \lim_{x\to+\infty} x^{\alpha+1}\exp(-x) = 0, \textit{for all } \alpha \in \mathbb{R}.$$

Since $\int_{1}^{+\infty} x^{-2}dx$ *converges, we deduce that* $\int_{1}^{+\infty} x^{\alpha-1}\exp(-x)dx$ *converges, for all* $\alpha \in \mathbb{R}$.
Finally $\int_{0}^{+\infty} x^{\alpha-1}\exp(-x)dx$ *converges if and only if* $\alpha > 0$.

Theorem 1.3.1. *The function* Γ *having the following property:*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \text{ for all } \alpha > 0. \tag{1.37}$$

In particular $\Gamma(n+1) = n!$, *for all* $n \in \mathbb{N}$.

Proof. An integration by parts gives us:

$$\begin{aligned} \Gamma(\alpha+1) &= \int_{0}^{+\infty} x^{\alpha}\exp(-x)dx \\ &= -\lim_{t\to+\infty}[x^{\alpha}\exp(-x)]_0^t + \alpha\int_{0}^{+\infty} x^{\alpha-1}\exp(-x)dx = \alpha\Gamma(\alpha). \end{aligned}$$

On the other hand $\Gamma(0) = \int_{0}^{+\infty} \exp(-x)dx = 1$, we deduce that $\Gamma(n+1) = n!$, for all $n \in \mathbb{N}$. □

Remak 1.2. *The function* Γ *can be extended to a function defined on the set of negative real numbers, except for* $\alpha = 0, -1, -2, -3, ...$
Indeed, from the relation (1.37), we can write:

$$\Gamma(\alpha-1) = \frac{\Gamma(\alpha)}{\alpha-1}, \quad -1 < \alpha-1 < 0.$$

$$\Gamma(\alpha-2) = \frac{\Gamma(\alpha-1)}{\alpha-2}, \quad -2 < \alpha-2 < -1.$$

In this way, we can find:

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}, \qquad , -n < \alpha < -(n-1).$$

Figure 1.1 shows the graph of Euler's gamma function.

Theorem 1.3.2. *The function* Γ *is infinitely derivable on* $\mathbb{R}_+^*$ *and furthermore, we have:*

$$\Gamma^{(n)}(\alpha) = \int_{0}^{+\infty} \ln^n(x)x^{\alpha-1}\exp(-x)dx. \tag{1.38}$$

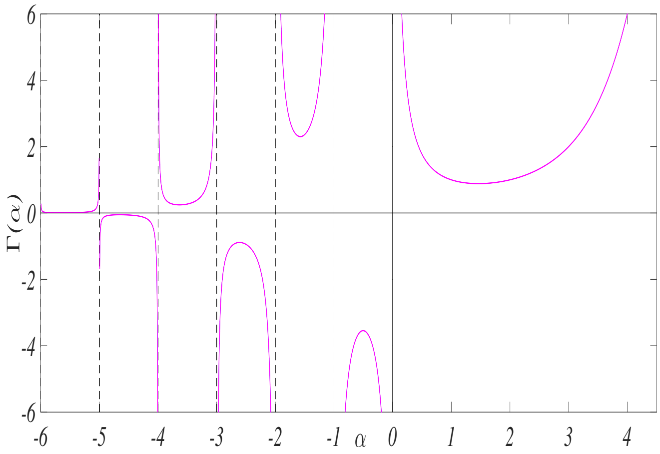

Figure 1.1: Graph of the gamma function.

Proof. Let us set $f(\alpha, x) = x^{\alpha-1}\exp(-x)$, $(\alpha, x) \in (\mathbb{R}_+^*)^2$.
For all $n \in \mathbb{N}$, f is of class C^n on $(\mathbb{R}_+^*)^2$ and furthermore, we have:

$$\frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x) = \ln^n(x) x^{\alpha-1}\exp(-x). \tag{1.39}$$

Consider any compact interval $[a, b]$ of $]0, +\infty[$. It suffices to show that the integral $\int_0^{+\infty} \frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)dx$ is uniformly convergent on any segment $[a, b]$. Indeed, for all $x \in]0, 1]$, the function $\alpha \mapsto x^{\alpha-1}$ is decreasing, so for all $\alpha \in [a, b]$, we have $0 < f(\alpha, x) \leq x^{a-1}$, and for all $x > 1$, the function $\alpha \mapsto x^{\alpha-1}$ is increasing, so for all $\alpha \in [a, b]$, we have $0 < f(\alpha, x) \leq x^{b-1}\exp(-x)$.
In the neighborhood of 0, we then have

$$\left|\frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)\right| = \left|\ln^n(x)\right| f(\alpha, x) = O\left(\frac{1}{x^{1-\frac{a}{2}}}\right), \tag{1.40}$$

and in the neighborhood of infinity,

$$\left|\frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)\right| = \left|\ln^n(x)\right| f(\alpha, x) = O\left(\frac{1}{x^2}\right). \tag{1.41}$$

Since the two integrals $\int_0^1 \frac{dx}{x^{1-\frac{a}{2}}}$ and $\int_1^{+\infty} \frac{dx}{x^2}$ are convergent, the Weierstrass criterion shows that the integral $\int_0^{+\infty} \frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)dx$ is also uniformly convergent on any segment $[a, b]$, which completes the establishment that Γ is of class C^n on $\mathbb{R}_+^*$ for all $n \in \mathbb{N}$, so that Γ is of class $C^{+\infty}$ on $\mathbb{R}_+^*$. □

1.3.2 Euler's Beta function

Definition 1.3.2. *We call Euler's beta function the special function B defined by:*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx, \text{ for } \alpha,\ \beta > 0. \tag{1.42}$$

Remak 1.3. *The Euler function beta defined by the relation (1.42) is well defined on* $(\mathbb{R}_+^*)^2$.

In the neighborhood of zero, $x^{\alpha-1} (1-x)^{\beta-1} \overset{V(0)}{\sim} x^{\alpha-1}$.

Since $\int_0^{\frac{1}{2}} x^{\alpha-1} dx$ *converges if and only if* $\alpha > 0$, *we deduce that* $\int_0^{\frac{1}{2}} x^{\alpha-1} (1-x)^{\beta-1} dx$ *converges if and only if* $\alpha > 0$ *and* $\beta \in \mathbb{R}$.

In the neighborhood of 1, $x^{\alpha-1} (1-x)^{\beta-1} \overset{V(0)}{\sim} x^{\beta-1}$.

Since $\int_{\frac{1}{2}}^{1} x^{\alpha-1} dx$ *converges if and only if* $\beta > 0$, *we deduce that* $\int_{\frac{1}{2}}^{1} x^{\alpha-1} (1-x)^{\beta-1} dx$ *converges if and only if* $\beta > 0$ *and* $\alpha \in \mathbb{R}$.

Finally $\int_0^1 x^{\alpha-1} \exp(-x) dx$ *converges if and only if* $\alpha > 0$ *and* $\beta > 0$.

Theorem 1.3.3. *(Symmetry) The function B is symmetric, that is, for all* $\alpha, \beta > 0$, *we have:*

$$B(\alpha, \beta) = B(\beta, \alpha). \tag{1.43}$$

Proof. Indeed, by making the change of variable $x = 1 - t$, we immediately find the result. □

Theorem 1.3.4. *(Another formula for the beta function) The beta function can be represented by the following formula:*

$$B(\alpha, \beta) = \int_0^{+\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt. \tag{1.44}$$

Proof. Indeed, by performing the change of variable $x = \dfrac{t}{1+t}$, we immediately find the result. □

1.3.3 Relationship between gamma and beta functions

Theorem 1.3.5. *1) For all $\alpha, \beta > 0$, we have:*

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.45}$$

2) The functions B and Γ verify the following Euler reflection formula:

$$B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}, \ \alpha \in \left]0, 1\right[. \tag{1.46}$$

Proof. By making the change of variable $x = (1+z)y$ $(z > 0)$ in the relation (1.36), we find:

$$\frac{\Gamma(\alpha+\beta)}{(1+z)^{\alpha+\beta}} = \int_0^{+\infty} y^{\alpha+\beta-1} \exp\left(-(1+z)\,y\right) dy, \ \alpha, \beta, z > 0. \tag{1.47}$$

Multiplying both sides of the equality (1.47) by $z^{\alpha-1}$, then integrating with respect to z from 0 to $+\infty$, we find:

$$\Gamma(\alpha+\beta) \int_0^{+\infty} \frac{z^{\alpha-1}}{(1+z)^{\alpha+\beta}} dz = \int_0^{+\infty} z^{\alpha-1} dz \left(\int_0^{+\infty} y^{\alpha+\beta-1} \exp\left(-(1+z)\,y\right) dy \right), \text{ for all } \alpha, \beta, z > 0, \tag{1.48}$$

or in an equivalent manner:

$$\begin{aligned}
\Gamma(\alpha+\beta) \times B(\alpha, \beta) &= \int_0^{+\infty} z^{\alpha-1} \left(\int_0^{+\infty} y^{\alpha+\beta-1} \exp\left(-(1+z)\,y\right) dy \right) dz \\
&= \int_0^{+\infty} y^{\alpha+\beta-1} \exp(-y) \left(\int_0^{+\infty} z^{\alpha-1} \exp(-zy)\, dz \right) dy \\
&= \int_0^{+\infty} y^{\alpha+\beta-1} \exp(-y) \frac{\Gamma(\alpha)}{y^{\alpha}} dy = \Gamma(\alpha) \int_0^{+\infty} y^{\beta-1} \exp(-y)\, dy \\
&= \Gamma(\alpha)\Gamma(\beta)
\end{aligned}$$

2) The equality $B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$ follows immediately from the relation (1.45) by setting $\beta = 1 - \alpha$.

Using the second formula of the beta function, we find:

$$B(\alpha, 1-\alpha) = \int_0^{+\infty} \frac{t^{\alpha-1}}{1+t}dt \tag{1.49}$$

Let us now calculate the integral (1.49) by the theorem of residues. We then define the following path, for $f(z) = \dfrac{z^{\alpha-1}}{1+z}$ and $0 < \epsilon < 1 < R$:

a) C_ϵ the half circle of radius ϵ on the half plane $\Re(z) < 0$.

b) The two segments $S^{\pm}_{\epsilon,R} = \left\{\pm i\epsilon,\ \ \pm i\epsilon + \sqrt{R^2-\epsilon^2}\right\}$.

c) The arc of a circle $\Gamma_{\epsilon,R} = \left\{R\exp(i\theta),\ \theta \in \left[\arctan\dfrac{\epsilon}{\sqrt{R^2-\epsilon^2}}, 2\pi - \arctan\dfrac{\epsilon}{\sqrt{R^2-\epsilon^2}}\right]\right\}$.

Let us choose ϵ and R such that $z_0 = -1$ is in the loop. Using the residue theorem gives us:

$$\int_{C_\epsilon} f(z)dz + \int_{S^-_{\epsilon,R}} f(z)dz + \int_{\Gamma_{\epsilon,R}} f(z)dz + \int_{S^+_{\epsilon,R}} f(z)dz = 2\pi i \times \mathrm{Res}\,(f,-1)\,. \tag{1.50}$$

Passing to the limit, when $\epsilon \to 0$ and $R \to +\infty$, it comes by Jordan's lemma that:

$$\lim_{\substack{\epsilon\to 0\\ R\to+\infty}} \int_{C_\epsilon} f(z)dz + \int_{\Gamma_{\epsilon,R}} f(z)dz = 0 + 0 = 0. \tag{1.51}$$

On the other hand, for all $t > 0$, we have:

$$\lim_{\epsilon\to 0}(t+i\epsilon)^{1-\alpha} = t^{1-\alpha} \text{ et } \lim_{\epsilon\to 0}(t-i\epsilon)^{1-\alpha} = t^{1-\alpha}\exp(-2\pi i\alpha). \tag{1.52}$$

So:

$$\lim_{\epsilon\to 0}(t+i\epsilon)^{\alpha-1} = t^{\alpha-1} \text{ et } \lim_{\epsilon\to 0}(t-i\epsilon)^{\alpha-1} = t^{\alpha-1}\exp(2\pi i\alpha). \tag{1.53}$$

From (1.50), (1.51) and (1.53), we can then write:

$$\exp(2\pi i\alpha)\int_{+\infty}^{0} \frac{z^{\alpha-1}}{1+z}dz + \int_0^{+\infty} \frac{z^{\alpha-1}}{1+z}dz = 2\pi i \times Res\,(f,-1)\,. \tag{1.54}$$

We deduce:

$$\begin{aligned}(1-\exp(2\pi i\alpha))\int_0^{+\infty} \frac{z^{\alpha-1}}{1+z}dz &= 2\pi i \times Res\,(f,-1) = 2\pi i \times \lim_{z\to-1} z^{\alpha-1}\\ &= 2\pi i \times \lim_{z\to-1}\frac{1}{z^{1-\alpha}} = -\exp(i\pi\alpha),\end{aligned}$$

so, after simplification, we find:

$$\int_0^{+\infty} \frac{z^{\alpha-1}}{1+z}dz = \frac{\pi}{\sin(\alpha\pi)},\ \alpha > 0.$$

$\square$

Example 1.3.1. *Consider the integral* $I = \int_0^{+\infty} \frac{t^{\frac{2}{3}}}{(1+t)^3}dt$

We have:

$$\begin{aligned} I &= \int_0^{+\infty} \frac{t^{\frac{5}{3}-1}}{(1+t)^{\frac{5}{3}+\frac{4}{3}}}dt = B\left(\frac{5}{3}\frac{4}{3}\right) = \frac{\Gamma(\frac{5}{3})\Gamma(\frac{4}{3})}{\Gamma(3)} \\ &= \frac{\Gamma(\frac{2}{3}+1)\Gamma(\frac{1}{3}+1)}{2} = \frac{\frac{2}{3}\times\Gamma(\frac{2}{3})\times\frac{1}{3}\Gamma(\frac{1}{3})}{2} \\ &= \frac{1}{9}\frac{\Gamma(1-\frac{1}{3})\times\Gamma(\frac{1}{3})}{\Gamma(1-\frac{1}{3}+\frac{1}{3})} = B\left(\frac{1}{3}, 1-\frac{1}{3}\right) \\ &= \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}. \end{aligned}$$

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