Solution of Series of Exercises 05

Solution of Exercise 1

Let us study whether the following improper integrals make sense and give their possible values.

1. $I_n = \int_0^{+\infty} x^n \exp(-x)dx$, $n \in \mathbb{N}$. We have:

$$x^n \exp(-x) = O\left(\exp(-\frac{x}{2})\right), \text{ when } x \to +\infty.$$

Since $\int_0^{+\infty} \exp(-\frac{x}{2})dx$ converges, it follows that $\int_0^{+\infty} x^n \exp(-x)dx$ converges.
Let's calculate its value:
We perform integration by parts.
Let

$$f(x) = x^n \text{ and } \dot{g}(x) = \exp(-x),$$

we then have:

$$\dot{f}(x) = nx^{n-1} \text{ and } g(x) = -\exp(-x).$$

Consequently:

$$\begin{aligned} I_n &= \int_0^{+\infty} x^n \exp(-x)dx \\ &= \lim_{y\to+\infty} \left[-nx^{n-1}\exp(-x)\right]_0^y + n\int_0^{+\infty} x^{n-1}\exp(-x)dx \\ &= nI_{n-1}. \end{aligned}$$

By recurrence, we find $I_n = n!I_0$.
Since $\int_0^{+\infty} \exp(-x)dx = 1$, we have $I_n = n!$.

Solution of Exercise 1

2. $J_n = \int_0^1 \ln^n(x)dx, \ n \in \mathbb{N}.$

We always perform integration by parts. Let

$$\dot{f}(x) = 1 \text{ and } g(x) = \ln^n(x),$$

we find:

$$f(x) = x \text{ and } \dot{g}(x) = n\frac{\ln^{n-1}(x)}{x}.$$

Consequently:

$$\begin{aligned} J_n &= \int_0^1 \ln^n(x)dx \\ &= \lim_{y\to 0}\left[x\ln^n(x)\right]_y^1 - n\int_0^1 \ln^{n-1}(x)dx \\ &= -nJ_{n-1}. \end{aligned}$$

By recurrence, we find $J_n = (-1)^n n! J_0$.

Since $J_0 = 1$, we have $J_n = (-1)^n n!$, and the integral under consideration converges.

3. $I = \int_0^2 \frac{dx}{\sqrt{(2x-x^2)}}$. Let $f(x) = \frac{1}{\sqrt{(2x-x^2)}}$.

It is sufficient to study the convergence of the two integrals $\int_0^1 \frac{dx}{\sqrt{(2x-x^2)}}$ and $\int_1^2 \frac{dx}{\sqrt{(2x-x^2)}}$.

In the neigborhood of 0^+, we have:

$$f(x) = O\left(\frac{1}{\sqrt{x}}\right), \text{ as } x \to 0^+.$$

$\int_0^1 \frac{dx}{\sqrt{x}}$ is therefore the Riemann integral which converges, so the integral $\int_0^1 \frac{dx}{\sqrt{(2x-x^2)}}$ is convergent.

In the neigborhood of 2^-, we have:

$$f(x) = O\left(\frac{1}{\sqrt{2-x}}\right), \text{ as } x \to 2^-.$$

Solution of Exercise 1

$\int_1^2 \frac{dx}{\sqrt{2-x}}$ is therefore the Riemann integral which converges, so the integral $\int_1^2 \frac{dx}{\sqrt{(2x-x^2)}}$ is convergent.

We can therefore deduce that $\int_1^2 \frac{dx}{\sqrt{(2x-x^2)}}$ converges.

Let's calculate its value:

Let's use the change of variable

$$y \in \left]0, \frac{\pi}{2}\right[\mapsto x = g(y) = 2\sin^2 y,$$

The function g from $\left]0, \frac{\pi}{2}\right[$ to $]0, 2[$, is an increasing bijection of class C^1. We get:

$$\begin{aligned} \int_0^2 \frac{dx}{\sqrt{(2x-x^2)}} &= \int_0^{\frac{\pi}{2}} \frac{4\sin y\cos y}{2\sin y\cos y} dy \\ &= 2\int_0^{\frac{\pi}{2}} dy = \pi \end{aligned}$$

Solution of Exercise 2

Let's study the convergence of the following integrals:

1. $I_1 = \int_0^{\frac{1}{2}} \frac{\sin x}{x^\alpha |\ln x|^\beta} dx$. Let $f(x) = \frac{\sin x}{x^\alpha |\ln x|^\beta}$. This function is positive on $\left]0, \frac{1}{2}\right[$, equivalent in the neighborhood of 0^+ to $\frac{1}{x^{\alpha-1} |\ln x|^\beta}$.
This is then a Bertrand integral that converges if and only if $\alpha < 2$ and $\beta \in \mathbb{R}$, or $\alpha = 2$ and $\beta > 1$.
2. $I_2 = \int_0^{\frac{1}{2}} \frac{\ln(1 + x\sin x)}{|x \ln x|^\delta} dx$. Let $f(x) = \frac{\ln(1 + x\sin x)}{|x \ln x|^\delta}$. This function is positive on $\left]0, \frac{1}{2}\right[$, equivalent in the neighborhood of 0^+ to $\frac{x^2}{x^\delta \left|\ln^\delta x\right|} = \frac{1}{x^{\delta-2} \left|\ln^\delta x\right|}$.
This is then a Bertrand integral that converges if and only if $\delta - 2 < 1$ and $\delta \in \mathbb{R}$, or $\delta - 2 = 1$ and $\delta > 1$, i.e; $\delta \leq 3$.
3. $I_3 = \int_0^{\frac{1}{2}} \frac{\arctan(x^2)}{x^3 |\ln x|} dx$. Let $f(x) = \frac{\arctan(x^2)}{x^3 |\ln x|}$. This function is positive on $\left]0, \frac{1}{2}\right[$, equivalent in the neighborhood of 0^+ to $\frac{x^2}{x^3 |\ln x|} = \frac{1}{x |\ln x|}$.
This is then a Bertrand integral that diverges.
4. $I_4 = \int_1^2 \frac{|x-1|^\alpha}{|\ln x|^\beta} dx$. Let $f(x) = \frac{|x-1|^\alpha}{|\ln x|^\beta}$. This function is positive on $]1, 2[$, equivalent in the neighborhood of 1 to $\frac{|x-1|^\alpha}{|x-1|^\beta} = \frac{1}{|x-1|^{\beta-\alpha}}$.

Solution of Exercise 2

A comparison with the Riemann integral confirms that this integral converges if and only if $\beta - \alpha < 1$.

5. $I_4 = \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$. Let $f(x) = \frac{1}{\sqrt{(x-1)(2-x)}}$. This function is positive on $]1, 2[$, equivalent in the neighborhood of 1^+ to $\frac{1}{\sqrt{(x-1)}}$. This is then a Riemann integral that converges.
In the neighborhood of 2^-, f is equivalent to $\frac{1}{\sqrt{(2-x)}}$. This is then a convergent Riemann integral. We therefore deduce that $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$ is convergent.

Solution of Exercise 3

Let us study the absolute convergence and semi-convergence of the following generalizd integrals.:

1. $I_1 = \int_1^{+\infty} \frac{\sin x}{x^\theta} dx$, $\theta \in \mathbb{R}_+^*$. Let $f(x) = \frac{\sin x}{x^\theta}$.

If $\theta > 1$, $|f(x)| \leq \frac{1}{x^\theta}$ A comparison with the Riemann integral confirms that this integral converges.

So $I_1 = \int_1^{+\infty} \frac{\sin x}{x^\theta} dx$ is absolutely convergent, for all $\theta > 1$.

Let $0 < \theta < 1$. So, for all $x \geq 1$, we have

$$|f(x)| \geq \frac{\sin^2 x}{x^\theta} = \frac{1}{2x^\theta} - \frac{\cos 2x}{x^\theta}.$$

Since $\int_1^{+\infty} \frac{1}{x^\theta} dx$ is the Riemann integral, which diverges (because $\theta < 1$) and $\int_1^{+\infty} \frac{\cos 2x}{x^\theta} dx$ converges (according to Abel criterion), so $\int_1^{+\infty} \left(\frac{1}{2x^\theta} - \frac{\cos 2x}{x^\theta}\right) dx$ diverges , it follows that $I_1 = \int_1^{+\infty} \frac{\sin x}{x^\theta} dx$ does not converge absolutely.

On the other hand, this integral is convergent (according to Abel criterion), which shows the semi-convergence of this integral, for all $0 < \theta < 1$.

2. $I_2 = \int_1^{+\infty} \frac{\cos x}{\sqrt{x} + \cos x} dx$. An asymptotic development of order 2 to the function $\frac{1}{1 + \frac{\cos x}{\sqrt{x}}}$ in the neigborhood of $+\infty$ allows us to write:

Solution of Exercise 3

$$\begin{aligned}
\frac{\cos x}{\sqrt{x} + \cos x} &= \frac{\cos x}{\sqrt{x}} \times \frac{1}{1 + \frac{\cos x}{\sqrt{x}}} \\
&= \frac{\cos x}{\sqrt{x}} \left(1 - \frac{\cos x}{\sqrt{x}} + \frac{\cos^2 x}{x}\left(1 + \epsilon(\frac{1}{\sqrt{x}})\right)\right), \epsilon(\frac{1}{\sqrt{x}}) \to 0 \text{ as } x \to +\infty \\
&= \frac{\cos x}{\sqrt{x}} - \frac{\cos^2 x}{x} + \frac{\cos^3 x}{x\sqrt{x}}\left(1 + \epsilon(\frac{1}{\sqrt{x}})\right), \epsilon(\frac{1}{\sqrt{x}}) \to 0 \text{ as } x \to +\infty.
\end{aligned}$$

The first two integrals are semi-convergent and the third is absolutely convergent. We deduce that $I_2 = \int_1^{+\infty} \frac{\cos x}{\sqrt{x} + \cos x} dx$ is semi-convergent.

Solution of Exercise 4

1. Let's show that $F(\alpha)=\int\limits_0^{+\infty}\frac{\cos(\alpha x)}{1+x^2}dx$ is absolutely.

Since $\left|\frac{\cos(\alpha x)}{1+x^2}\right|\le\frac{1}{1+x^2}$, and $\int\limits_0^{+\infty}\frac{1}{1+x^2}dx=\frac{\pi}{2}$, which is convergent, then the considered integral is absolutely convergent.

2. Let $\alpha>0$.

We perform integration by parts on the segment $[0,y]$ by setting:

$$\begin{aligned} f(x) &= \frac{1}{1+x^2},\ \dot{f}(x)=\frac{2x}{(1+x^2)^2} \\ \text{et } \dot{g}(x) &= \cos(\alpha x),\ g(x)=\frac{\sin(\alpha x)}{\alpha}, \end{aligned}$$

we then have:

$$\int\limits_0^{y}\frac{\cos(\alpha x)}{1+x^2}dx=\frac{1}{\alpha}\left(\left[\frac{\sin(\alpha x)}{1+x^2}\right]_0^y-\int\limits_0^{y}\frac{2x\sin(\alpha x)}{(1+x^2)^2}dx\right).$$

We take $y\to+\infty$, we find:

$$F(\alpha)=\frac{1}{\alpha}\int\limits_0^{+\infty}\frac{2x\sin(\alpha x)}{(1+x^2)^2}dx.$$

Solution of Exercise 4

So

$$\begin{aligned} |F(\alpha)| &= \frac{1}{\alpha}\int\limits_0^{+\infty}\frac{2x\,|\sin(\alpha x|)}{(1+x^2)^2}dx \\ &\le \frac{1}{\alpha}\int\limits_0^{+\infty}\frac{2x\,|\sin(\alpha x|)}{(1+x^2)^2}dx\le\frac{1}{\alpha}\int\limits_0^{+\infty}\frac{2x}{(1+x^2)^2}dx \\ &= \frac{1}{\alpha}\lim_{t\to+\infty}[\frac{1}{1+x^2}]_0^t=\frac{1}{\alpha}. \end{aligned}$$

We can therefore deduce that $\lim_{\alpha\to+\infty}F(\alpha)=0$.

Solution of Exercise 5

Let P et Q two polynomials such that: $\deg(P) < \deg(Q)$.

Let us study the convergence of the following generalized integral:

$$I = \int_a^{+\infty} \frac{P(x)}{Q(x)} \cos(x) dx.$$

1. If $\deg(P) = \deg(Q) - 1$:

An asymptotic expansion in the neighborhood of $+\infty$ allows us to write:

$$\frac{P(x)}{Q(x)} \cos(x) = \frac{\alpha}{x} \cos(x) + \frac{\cos(x)}{x^n} (\beta + \epsilon(x)), \ \epsilon(x) \to 0 \text{ as } x \to +\infty,$$

where α et β are reals, $n \geq 2$ is a positive integer .

The first integral is semi-convergent and the second is absolutely convergent. We deduce that $\int_a^{+\infty} \frac{P(x)}{Q(x)} \cos(x) dx$ is semi-convergent.

2. 1. If $\deg(P) \leq \deg(Q) - 2$:

An asymptotic expansion in the neighborhood of $+\infty$ allows us to write:

$$\frac{P(x)}{Q(x)} \cos(x) = \frac{\alpha}{x^n} \cos(x) (1 + \epsilon(x)), \ n \geq 2 \text{ and } \epsilon(x) \to 0 \text{ as } x \to +\infty.$$

We then obtain an absolutely convergent integral.