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Generalized Integrals

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Contents

Chapter 1

Generalized (improper) integrals

This chapter mainly consists in generalizing the notion of Riemann integrals to unbounded functions defined on intervals that are not necessarily bounded.

1.1 Convergence of generalized integrals

Definition 1.1.1. *Let $f : [a, b[$(or $b = +\infty$)$\to \mathbb{R}$ be a locally integrable function on $[a, b[$ (i.e. its restriction to each compact of $[a, b[$ is Riemann-integrable), and let F be the function defined on $[a, b[$ by:*

$$F(x) = \int_a^x f(t)dt, \text{for all } x \in [a, b[\,. \tag{1.1}$$

The generalized integral $\int_a^b f(t)dt$ is convergent, if and only if $\lim_{x\to b^-} F(x) = l \in \mathbb{R}$. Otherwise, the integral $\int_a^b f(t)dt$ is divergent.

Remak 1.1. *Let $f : [a, b[$(or $b = +\infty$)$\to \mathbb{R}$ be a locally integrable function on $[a, b[$. For all $c \in [a, b[$, we can write:*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt. \tag{1.2}$$

Since $\int_a^c f(t)dt$ always converges (Riemann integral), the generalized integrals $\int_a^b f(t)dt$ and $\int_c^b f(t)dt$ are therefore of the same nature.

Definition 1.1.2. *Now, let $f :]a, b]$(or $a = -\infty$)$\to \mathbb{R}$ be a locally integrable function on $]a, b]$ and let G be the function defined on $]a, b]$ by:*

$$G(x) = \int_x^b f(t)dt, \text{for all } x \in]a, b]. \tag{1.3}$$

The generalized integral $\int_a^b f(t)dt$ is convergent, if and only if $\lim_{x\to a^+} G(x) = l \in \mathbb{R}$. Otherwise, the integral $\int_a^b f(t)dt$ is divergent.

Definition 1.1.3. *Let f be a locally integrable function on $]a, b[$ where $a \in \mathbb{R}\cup\{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ and let $c \in]a, b[$.*
The generalized integral $\int_a^b f(t)dt$ is said to be convergent, if and only the two generalizedl integrals $\int_a^c f(t)dt$ and $\int_c^b f(t)dt$ are convergent. Otherwise, this integral is said to be divergent.

Example 1.1.1. *Consider the generalized integral $\int_0^{+\infty} \frac{dt}{t^2}$. Since:*

$$\int_x^1 \frac{dt}{t^2} = -1 + \frac{1}{x} \to +\infty, \text{ when } x \to 0^+, \tag{1.4}$$

the integral $\int_0^1 \frac{dt}{t^2}$ diverges, however

$$\int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x} \to 1, \text{ when } x \to +\infty. \tag{1.5}$$

So the integral $\int_1^{+\infty} \frac{dt}{t^2}$ converges. We deduce that the integral $\int_0^{+\infty} \frac{dt}{t^2}$ diverges.

Proposition 1.1.1. *Let f and g be two locally integrable functions on $[a, b[$.*
1. If the integrals $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ converge, then the integral $\int_a^b (f(t) + g(t))\, dt$ is also convergent, and moreover:

$$\int_a^b (f(t) + g(t))\, dt = \int_a^b f(t)dt + \int_a^b g(t)dt. \tag{1.6}$$

On the other hand, if the integral $\int_a^b f(t)dt$ concerges and $\int_a^b g(t)dt$ diverges, then $\int_a^b (f(t) + g(t))\, dt$ diverges.
2. Let $\alpha \in \mathbb{R}$, then the integral $\int_a^b \alpha f(t)dt$ converges, and moreover:

$$\int_a^b \alpha f(t)dt = \alpha \int_a^b f(t)dt. \tag{1.7}$$

Proof. The proof of this proposition is based on the linearity of Riemann integrals and on the theorems on the sum and product of limits. □

1.2 Integration formulas for generalized integrals

The following two theorems are very useful in the study of generalized integrals:

1.2.1 Integration by parts

Theorem 1.2.1. *Let f and g be two functions of class C^1 on $]a,b[$. If the integral $\int_a^b \dot{f}(t)g(t)dt$ converges and if the function fg has a right-hand limit at a and a left-hand limit at b, then the integral $\int_a^b f(t)\dot{g}(t)dt$ converges, and moreover:*

$$\int_a^b \dot{f}(t)g(t)dt = \left(\lim_{x\to b^-} f(x)g(x) - \lim_{x\to a^+} f(x)g(x)\right) - \int_a^b \dot{g}(t)f(t)dt \tag{1.8}$$

Proof. It is based on the fact that the function fg is the primitive of the function $\dot{f}g + f\dot{g}$ on the compact interval $[x,y]$, $x,y \in]a,b[$ and that fg has a right-hand limit as $x \to a^+$ and has a left-hand limit as $y \to b^-$. □

1.2.2 Change of variables

Theorem 1.2.2. *Let f be a continuous function on $]a,b[$ and let φ be a bijective function of class C^1 on $]\alpha,\beta[$, verifying $a = \lim_{x\to\alpha^+} \varphi(x)$ and $b = \lim_{x\to\beta^-} \varphi(x)$. Then the integrals $\int_a^b f(t)dt$ and $\int_\alpha^\beta \dot{\varphi}(x)f(\varphi(x))dx$ are of the same nature, and moreover, if they are convergent, we have:*

$$\int_a^b f(t)dt = \int_\alpha^\beta \dot{\varphi}(x)f(\varphi(x))dx. \tag{1.9}$$

Proof. It is based on the change of variable $t = \varphi(x)$ and the fact that $a = \lim_{x\to\alpha^+} \varphi(x)$ et $b = \lim_{x\to\beta^-} \varphi(x)$.

□

1.3 Generalized integral of functions with constant sign

In the following, we are interested in the case where the functions f and g are locally integrable and with constant sign on the interval $[a, b[$ or $]a, b]$. In particular, we will state all the results in the case where the functions f and g are positive. If the functions f and g are negative, we will study the integral of the functions $-f$ and $-g$.

Theorem 1.3.1. *Let f be a positive and locally integrable function on $[a, b[$. Then the integral $\int_a^b f(t)dt$ converges if and only if the function*

$$x \mapsto F(x) = \int_a^x f(t)dt, x \in [a, b[\,, \tag{1.10}$$

is bounded above $[a, b[$, and moreover, we have

$$\textit{for all } x \in [a, b[\,; F(x) \leq \int_a^b f(t)dt, \tag{1.11}$$

Proof. Since f is positive, the function F is therefore increasing. But for $\lim_{x\to b^-} F(x)$ to exist, it is necessary and sufficient that F be bounded above, and we have:

$$F(x) = \int_a^x f(t)dt \leq \int_a^b f(t)dt. \tag{1.12}$$

□

1.3.1 Comparison theorem

Theorem 1.3.2. *Let f and g be two positive and locally integrable functions on $[a, b[$ or $]a, b]$ verifying:*

$$0 \leq f(t) \leq g(t). \tag{1.13}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, then the integral $\int_a^b f(t)dt$ is also convergent, and we have:

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt. \tag{1.14}$$

2. If the integral $\int_a^b f(t)dt$ is divergent, then the integral $\int_a^b g(t)dt$ is also divergent.

Proof. To prove this theorem, suppose for example that f and g are defined on $[a, b[$.

For all $x \in [a, b[$, we have

$$F(x) = \int_a^x f(t)dt \leq \int_a^x g(t)dt = G(x). \tag{1.15}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, G is bounded above, F is also is bounded above, hence the result.

2. If the integral $\int_a^b f(t)dt$ diverges, this means that $\lim_{x \to b^-} F(x) = +\infty$, so $\lim_{x \to b^-} G(x) = +\infty$ and the integral $\int_a^b g(t)dt$ diverges. □

Corollary 1.3.1. *Let f and g be two positive and locally integrable functions on $[a, b[$ verifying:*

$$f(t) = O(g(t)), \text{ when } t \to b^-. \tag{1.16}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, then the integral $\int_a^b f(t)dt$ is also convergent

2. If the integral $\int_a^b f(t)dt$ is divergent, then the integral $\int_a^b g(t)dt$ is also divergent.

Proof. By definition: $f(t) = O(g(t))$, when $t \to b^-$, if :

$$\exists t_0 \in [a, b[\text{ , and } \exists c > 0, \text{ such that } \forall t \in [t_0, b[\text{ , we have } f(t) \leq cg(t). \tag{1.17}$$

The rest of the proof follows immediately from Theorem 1.3.2. □

Corollary 1.3.2. *Let f and g be two positive and locally integrable functions on $[a, b[$ verifying:*

$$\lim_{t \to b^-} \frac{f(t)}{g(t)} = 0. \tag{1.18}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, then the integral $\int_a^b f(t)dt$ is also convergent

2. If the integral $\int_a^b f(t)dt$ is divergent, then the integral $\int_a^b g(t)dt$ is also divergent.

Proof. By definition $\lim_{t \to b^-} \dfrac{f(t)}{g(t)} = 0$ if and only if:

$$\forall \epsilon > 0, \exists t_0 \in [a, b[\text{ , such that } \forall t \in [t_0, b[\text{ , } \frac{f(t)}{g(t)} < \epsilon. \tag{1.19}$$

The rest of the proof follows immediately from the Theorem 1.3.2, taking $\epsilon = 1$. □

1.3.2 Generalized integral of two positive equivalent functions

Definition 1.3.1. *Recall that two functions f and g are equivalent in the neighborhood of a point t_0, if and only if:*

$$\lim_{t\to t_0}\frac{f(t)}{g(t)}=1. \tag{1.20}$$

Theorem 1.3.3. *Let f and g be two positive functions equivalent to the end of the integration interval $[a,b[$ or $]a,b]$. Then the two integrals $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ are of the same nature.*

Proof. Suppose for example that f and g are defined on $[a,b[$.
By definition, $\lim_{t\to b}\dfrac{f(t)}{g(t)}=1$, if and only if:

$$\forall\epsilon>0, \exists\alpha\in[a,b[\,,\text{ such that } \forall t\in[\alpha,b[\,,(1-\epsilon)g(t)\le f(t)\le(1+\epsilon)g(t). \tag{1.21}$$

For $\epsilon=\dfrac{1}{2}$ fixed, we then have:

$$\forall t\in[\alpha,b[\,,\frac{1}{2}g(t)\le f(t)\le\frac{3}{2}g(t). \tag{1.22}$$

We can therefore apply the Theorem 1.3.2: If $\int_a^b g(t)dt$ converges, $\int_a^b f(t)dt$ also converges by the inequality on the right and if $\int_a^b f(t)dt$ converges, $\int_a^b g(t)dt$ also converges by the inequality on the left.
We make an analogous demonstration to show that if one of the integrals diverges, then the same is true for the other. □

Proposition 1.3.1. *(Riemann functions)*
1. Let f be a locally integrable function on $[1,+\infty[$. Then the Riemann integral $\int_1^{+\infty}\dfrac{dt}{t^\alpha}$ converges if and only if $\alpha>1$.
2. Now, let f be a locally integrable function on $]0,1]$. Then the Riemann integral $\int_0^1\dfrac{dt}{t^\alpha}$ converges if and only if $\alpha<1$.

Proof. 1. A simple calculation, we find:

$$\int_1^x\frac{dt}{t^\alpha}dt=\begin{cases}\dfrac{1}{1-\alpha}\left(\dfrac{1}{x^{\alpha-1}}-1\right), \text{ if } \alpha\neq1,\\ \ln x, \text{ if } \alpha=1.\end{cases}$$

These functions admit finite limits in the neighborhood of infinity only in the case where $\alpha > 1$.
2. Similarly, for $x \in]0,1[$

$$\int_x^1 \frac{dt}{t^\alpha} dt = \begin{cases} \frac{-1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}} - 1\right), \text{ if } \alpha \neq 1, \\ -\ln x, \text{ if } \alpha = 1. \end{cases} \tag{1.23}$$

These functions admit finite limits in the neighborhood of zero only in the case where $\alpha < 1$. □

Remak 1.2. *Using a change of variable $t \mapsto t - t_0$, allows us to also apply the preceding arguments to the functions $t \mapsto \frac{1}{(t-t_0)^\alpha}$ on the half-open intervals $]t_0, b]$ and $[b, +\infty[$, such that $b > t_0$.*

1.4 General convergence criteria

1.4.1 Cauchy criterion

Theorem 1.4.1. *Let f a locally integrable function on $[a,b[$. For the integral $\int_a^b f(t)dt$ to be convergent, it is necessary and sufficient that:*

$$\forall \epsilon > 0, \exists \delta > 0, \ \forall x, y \in [a,b[, b - \delta < x < y < b, \left|\int_x^y f(t)dt\right| < \epsilon. \tag{1.24}$$

Proof. It suffices to apply the Cauchy criterion to the function $x \mapsto F(x) = \int_a^x f(t)dt$ and the fact that $F(x) - F(y) = \int_x^y f(t)dt$. □

1.4.2 Abel-Dirichlet criterion

Lemma 1.4.1. *Let f be a function of class C^1, positive, decreasing on $[a,b[$ and tends to 0 when x tends to b, and let g be a continuous function on $[a,b[$ verifying the property:*

$$\exists M > 0, \forall x, y \in [a,b[, \left|\int_x^y g(t)dt\right| \leq M. \tag{1.25}$$

Then the integral $\int_a^b f(t)g(t)dt$ *converges, and for all* $x \in [a, b[$*, we have:*

$$\left|\int_x^b f(t)g(t)dt\right| \leq Mf(x). \tag{1.26}$$

Proof. By definition, we have:

$$\lim_{x \to b^-} f(x) = 0 \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0,\ \forall x \in [b-\delta, b[\,,\ f(x) < \frac{\epsilon}{M}. \tag{1.27}$$

Now let x be fixed in $[a, b[$ and let $G(y) = \int_x^y g(t)dt$.
By doing an integration by parts, we obtain:

$$\int_x^y f(t)g(t)dt = f(y)G(y) + \int_x^y (-\dot{f}(t))G(t)dt. \tag{1.28}$$

So for all $y \geq x \geq a$, we have:

$$\begin{aligned}
\left|\int_x^y f(t)g(t)dt\right| &\leq \left|f(y)G(y)\right| + \int_x^y \left|(-\dot{f}(t))G(t)\right| dt \\
&\leq Mf(y) + \int_x^y (-\dot{f}(t))Mdt \\
&= Mf(y) + Mf(x) - Mf(y) \\
&= Mf(x).
\end{aligned} \tag{1.29}$$

Using the inequality (1.27), we find:

$$\left|\int_x^y f(t)g(t)dt\right| < \epsilon. \tag{1.30}$$

The Cauchy criterion then allows us to assert that the integral $\int_a^b f(t)g(t)dt$ converges. Moreover, by making y tend towards b in the inequality (1.30), we obtain the inequality (1.26). □

1.5 Absolute convergence or semi-convergence

Let f be a locally integrable function on $[a, b[$.

Definition 1.5.1. *We say that* $\int_a^b f(t)dt$ *is absolutely convergent if* $\int_a^b \left|f(t)\right| dt$ *converges.*

Definition 1.5.2. *The integral $\int_a^b f(t)dt$ is said to be semi-convergent when it is convergent without being absolutely convergent.*

Theorem 1.5.1. *If the integral $\int_a^b f(t)dt$ converges absolutely, then $\int_a^b f(t)dt$ converges and moreover:*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b \left|f(t)\right| dt. \tag{1.31}$$

Proof. The proof proceeds using the Cauchy criterion and the fact that:

$$\left|\int_x^y f(t)dt\right| \leq \int_x^y \left|f(t)\right| dt,\ x, y \in [a, b[\tag{1.32}$$

□

Example 1.5.1. *Consider the function f defined on $\left[\frac{\pi}{2}, +\infty\right[$ by $f(x) = \frac{\sin t}{t^2}$. For all $t \geq \frac{\pi}{2}$, we have:*

$$\left|f(t)\right| \leq \frac{1}{t^2}. \tag{1.33}$$

Since the integral $\int_{\frac{\pi}{2}}^{+\infty} \frac{dt}{t^2}$ converges, we deduce that $\int_{\frac{\pi}{2}}^{+\infty} f(t)dt$ converges absolutely.

The following theorem is widely used in practice.

Theorem 1.5.2. *Let f be a locally integrable function on the interval $[a, +\infty[$, where $a > 0$, such that there exists $\alpha > 1$ verifying $\lim_{t\to+\infty} t^\alpha \left|f(t)\right| = 0$. Then the integral $\int_a^{+\infty} f(t)dt$ converges absolutely.*

Proof. By definition:

$$\lim_{t\to+\infty} t^\alpha \left|f(t)\right| = 0 \Leftrightarrow \forall \epsilon > 0, \exists A > a \text{ such that } \forall t \in [A, +\infty[\,,\ \left|f(t)\right| < \frac{\epsilon}{t^\alpha}.$$

Since the Riemann integral $\int_a^{+\infty} \frac{dt}{t^\alpha}$ converges, using the comparison theorem 1.3.2 shows the absolute convergence of the integral $\int_a^{+\infty} f(t)dt$. □

1.6 Generalized integrals and numerical series

In this section, we will give some results specifying the link between generalized integrals and numerical series.

Theorem 1.6.1. *Let f be a locally integrable function on the interval $[a, +\infty[$. Then the following properties are equivalent:*
1. The integral $\int_a^{+\infty} f(t)dt$ converges.
2. The numerical sequence with general term $F(x_n) = \int\limits_a^{x_n} f(x)dt$ converges, , where $(x_n)_n$ is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$.
3. The numerical series with term general $u_n = \int\limits_{x_n}^{x_{n+1}} f(x)dt$ converges, where $(x_n)_n$ is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$.

Proof. Let's show 1⇒ 2. For all $x \in [a, +\infty[$, the integral $\int_a^{+\infty} f(t)dt$ converges if and only:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \geq \delta, \text{ we have } \left| \int_a^{+\infty} f(t)dt - \int_a^{x} f(t)dt \right| = \left| \int_x^{+\infty} f(t)dt \right| < \epsilon. \tag{1.34}$$

Let now $(x_n)_n$ be a sequence of élements of $[a, +\infty[$ of limit $+\infty$, when $n \to +\infty$, i.e.

$$\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \ x_n \geq A \tag{1.35}$$

Let's take $A = \delta$, so for all $n \geq n_0$, and for all $x_n \geq \delta$, we have:

$$\left| \int_a^{+\infty} f(t)dt - F(x_n) \right| = \left| \int_a^{+\infty} f(t)dt - \int_a^{x_n} f(t)dt \right| = \left| \int_x^{+\infty} f(t)dt \right| < \epsilon. \tag{1.36}$$

Hence the sequence $F(x_n) = \int\limits_a^{x_n} f(n)dt$ converges.

Let us now show 2 ⇒ 1. Suppose that $F(x_n) = \int\limits_a^{x_n} f(x)dt$ converges and that the integral $\int_a^{+\infty} f(t)dt$ does not converge, we then have:

$$\exists \epsilon > 0, \forall \delta > 0, \text{ we can find } x > \delta, \text{ such that } \left| \int_a^{+\infty} f(t)dt - \int_a^{x} f(t)dt \right| \geq \epsilon. \tag{1.37}$$

As a result, we can find a sequence of elements (x_n) of $[a, +\infty[$verifying::

$$\left|\int_a^{+\infty} f(t)dt - \int_a^{x_n} f(t)dt\right| = \left|\int_a^{+\infty} f(t)dt - F(x_n)dt\right| \geq \epsilon,$$

which contradicts the hypothesis that $(F(x_n))_n$ converges.

Finally, let us show that 2 $\Leftrightarrow$ 3. For all $n \in \mathbb{N}$:

$$F(x_n) = \int_a^{x_n} f(n)dt = \int_a^{x_0} f(n)dt + \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(n)dt. \tag{1.38}$$

Therefore the numerical sequence $(F(x_n))_n$ converges if and only if the series of general term $u_n = \int_{x_n}^{x_{n+1}} f(n)dt$ converges. □

Corollary 1.6.1. *Similarly, when f is a locally integrable function on the interval $[a, +b[$, we can easily demonstrate that the integral $\int_a^b f(t)dt$ converges if and only if the numerical sequence with general term $F^*(x_n) = \int_a^{b-x_n} f(x)dt$ converges, where $(x_n)_n$ is a sequence of elements of $[a, +b[$ with limit 0, when $n \longrightarrow +\infty$.*

Theorem 1.6.2. *(Chauchy's theorem) Let $f :]0, +\infty[\rightarrow \mathbb{R}$ be a positive, continuous and decreasing definite function, then the series with positive terms $\sum_{n\geq 1} f(n)$ and the generalized integral $\int_1^{+\infty} f(x)dx$ are of the same nature.*

Proof. Let (S_n) be the sequence of partial sums of the series $\sum_{n\geq 1} f(n)$.

Since f is decreasing on $]0, +\infty[$, we can write:

$$\text{for all } k = 1, 2, ..., \ x \in]0, +\infty[\,, \text{ such that } k \leq x \leq k+1, \text{ we have } f(k+1) \leq f(x) \leq f(k). \tag{1.39}$$

Integrating over $[k, k+1]$, we find:

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k), \text{ for all } k = 1, 2, ... \tag{1.40}$$

By adding these last equalities, we obtain:

$$S_{n+1} - f(1) \leq \int_1^{n+1} f(x)dx \leq S_n. \tag{1.41}$$

* Suppose that $\int_1^{+\infty} f(x)dx$ converges, we can see then :

$$S_{n+1} - f(1) \leq \int_1^{n+1} f(x)dx \leq \int_1^{+\infty} f(x)dx. \tag{1.42}$$

Which shows that the sequence (S_{n+1}) is bounded above, and consequently the series $\sum_{n\geq 1} f(n)$ converges.
* Suppose that $\sum_{n\geq 1} f(n)$ converges. We know well that

$$n \leq x < n+1, \text{ for } x \geq 1, \tag{1.43}$$

where n represents the integer part of x. We then have:

$$\int_1^{x} f(x)dx \leq \int_1^{n+1} f(x)dx \leq S_n. \tag{1.44}$$

Since (S_n) is bounded above, the integral $\int_1^{x} f(x)dx$ is also, which ensures the existence of the integral $\int_1^{+\infty} f(x)dx$.
It also follows by contraposition that the divergence of the series $\sum_{n\geq 1} f(n)$ entails the divergence of the integral $\int_1^{+\infty} f(x)dx$.

□

1.7 Generalized integrals and numerical sequences

Theorem 1.7.1. *(Dominated convergence theorem) Let $(f_n)_n$ be a sequence of locally integrable functions on $[a,b[$, which converges uniformly locally on $[a,b[$ to a function f, and let g be a positive and locally integrable function on $[a,b[$ verifying the following two properties:*
1. The integral $\int_a^b g(t)dt$ converge.

2. *For all $n \in \mathbb{N}$ and for all $x \in [a, b[$, we have:*

$$\left|f_n(t)\right| \leq g(t). \tag{1.45}$$

So the two integrals $\int_a^b f_n(t)dt$ ($n \in \mathbb{N}$) and $\int_a^b f(t)dt$ converge absolutely, and we have:

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt. \tag{1.46}$$

Proof. The absolute convergence of the integral $\int_a^b f_n(t)dt$ proceeds from the inequality (1.45) and the use of the comparison theorem. Moreover, by making n tend to $+\infty$ in (1.45), we obtain:

$$\text{for all } x \in [a, b[\,, \ \left|f(t)\right| \leq g(t). \tag{1.47}$$

We then deduce the absolute convergence of the integral $\int_a^b f_n(t)dt$.
Let us now show that $\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.
By definition, the integral $\int_a^b g(t)dt$ converges if and only if:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in [b-\delta, b[\,, \text{ we have } \left|\int_x^b g(t)dt\right| < \frac{\epsilon}{3}. \tag{1.48}$$

On the other hand, the sequence $(f_n)_n$ converges uniformly locally to f on $[a, b[$, so it converges uniformly to f on any compact $[a, x]$, for any x fixed in $[b-\delta, b[$ i.e:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, \forall t \in [a, x] \ \ n \geq n_0, \ \left|f_n(t) - f(t)\right| < \frac{\epsilon}{3(x-a)}. \tag{1.49}$$

We then have:

$$\begin{aligned}
\left|\int_a^b f_n(t)dt - \int_a^b f(t)dt\right| &= \left|\int_a^b (f_n(t)dt - f(t))\, dt\right| \\
&= \left|\int_a^x (f_n(t)dt - f(t))\, dt + \int_x^b (f_n(t)dt - f(t))\, dt\right| \\
&\leq \int_a^x \left|f_n(t) - f(t)\right| dt + \int_x^b \left|f_n(t)\right| dt + \int_x^b \left|f(t)\right| dt \\
&\leq \int_a^x \left|f_n(t) - f(t)\right| dt + 2\int_x^b g(t)dt \\
&\leq \frac{\epsilon(x-a)}{3(x-a)} + \frac{2\epsilon}{3} = \epsilon.
\end{aligned}$$

□

1.8 Mean value theorems for integrals

1.8.1 First formula of the mean value

Theorem 1.8.1. *Let f and g be two functions verifying f integrable and with a constant sign on $[a,b]$ and g is continuous on the same segment, then there exists $c \in [a,b]$ verifying:*

$$\int_a^b f(t)g(t)dt = g(c)\int_a^b f(t)dt. \tag{1.50}$$

Proof. Let us place ourselves in the case where $f(x) > 0$, for all $x \in [a,b]$.
Let $m = \inf_{x\in[a,b]} g(x)$ and $M = \sup_{x\in[a,b]} g(x)$, we can then write:

$$m\int_a^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq M\int_a^b f(t)dt, \tag{1.51}$$

Hence:

$$m \leq \frac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt} \leq M. \tag{1.52}$$

That is to say that $\dfrac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt} \in \left[\inf_{x\in[a,b]} g(x), \sup_{x\in[a,b]} g(x)\right]$.
Since g is continuous, we can then find $c \in [a,b]$ verifying:

$$\frac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt} = g(c), \tag{1.53}$$

which completes the proof of the theorem. □

1.8.2 Second formula for the mean value

Theorem 1.8.2. *Let f be a function of class C^1, positive and decreasing on $[a,b]$, and let g be a continuous function on $[a,b]$, then there exists $c \in [a,b]$ verifying:*

$$\int_a^b f(t)g(t)dt = f(a)\int_a^c f(t)dt. \tag{1.54}$$

Proof. Let $G(x) = \int_a^x g(t)dt$.
By doing an integration by parts, we obtain:

$$\begin{aligned}\int_a^b f(t)g(t)dt &= f(x)G(x) + \int_a^b (-\dot{f}(t))G(t)dt \\ &= f(b)G(b) + \int_a^b (-\dot{f}(t))G(t)dt \text{ (because } G(a) = 0).\end{aligned} \quad (1.55)$$

Since f is increasing (i.e $\dot{f}(x) \leq 0$ on $[a, b]$) and $f(b) > 0$, we can then write:

$$f(b)\times m+\int_a^b (-\dot{f}(t))\times mdt \leq f(b)G(b)+\int_a^b (-\dot{f}(t))G(t)dt \leq f(b)\times M+\int_a^b (-\dot{f}(t))\times Mdt, \quad (1.56)$$

where $m = \inf_{x\in[a,b]} G(x)$ and $M = \sup_{x\in[a,b]} G(x)$, or in an equivalent manner:

$$mf(a) \leq \int_a^b f(t)g(t)dt \leq Mf(b). \quad (1.57)$$

That is to say that: $\dfrac{\int_a^b f(t)g(t)dt}{f(a)} \in \left[\inf_{x\in[a,b]} G(x), \sup_{x\in[a,b]} G(x)\right]$.
Since G is continuous, we can then find $c \in [a, b]$ verifying:

$$\frac{\int_a^b f(t)g(t)dt}{f(a)} = G(c) = \int_a^c g(t)dt, \quad (1.58)$$

which completes the proof of the theorem. □

1.9 Cauchy principal value

Definition 1.9.1. *Let f be a locally integrable function on $]-\infty, +\infty[$. The Cauchy principal value (or principal value of the divergent integral $\int_{-\infty}^{+\infty} f(t)dt$), denoted $V.P\left(\int_{-\infty}^{+\infty} f(t)dt\right)$ is the element of $\mathbb{R}$ defined by :*

$$V.P\left(\int_{-\infty}^{+\infty} f(t)dt\right) = \lim_{a\to\to+\infty} \int_{-a}^{a} f(t)dt) \text{ (when this limit exists).} \quad (1.59)$$

Definition 1.9.2. *Now, let f be a function, wich has a singular point $c \in]a, b[$. The Cauchy principal value (or principal value of the divergent integral $\int_a^b f(t)dt$),* *denoted* $V.P\left(\int_a^b f(t)dt\right)$ *is the element of* $\mathbb{R}$ *defined by:*

$$V.P\left(\int_a^b f(t)dt\right) = \lim_{\epsilon\to\to 0}\left(\int_a^{c-\epsilon} f(t)dt) + \int_{c+\epsilon}^b f(t)dt\right) \text{ (when this limit exists).} \tag{1.60}$$

Example 1.9.1. *As an example, we take the integral* $\int_0^2 \frac{dt}{t-1}$. *It is clear that this integral is divergent. We then have:*

$$\begin{aligned} V.P\left(\int_0^2 \frac{dt}{t-1}\right) &= \lim_{\epsilon\to\to 0}\left(\int_0^{1-\epsilon} \frac{dt}{t-1}) + \int_{1+\epsilon}^2 \frac{dt}{t-1}dt\right) \\ &= \lim_{\epsilon\to\to 0}\left(\ln\frac{\epsilon}{\epsilon} + \ln\frac{1}{1}\right) = 0. \end{aligned} \tag{1.61}$$

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