Solution of Series of Exercises 3

Solution of Exercise 1

Let's calculate the radius and the domain of convergence, as well as the sum of each of the following power series.:

1. $f_1(x) = \sum\limits_{n=0}^{+\infty} a^n x^n$, $a \neq 0$.

Radius of convergence:

We have $a_n = a^n$, which implies that:

$$R = \left[\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|\right]^{-1} = \frac{1}{a}.$$

Domain de convergence:

If $|x| = \dfrac{1}{a}$, we have $|a^n x^n| = 1 \nrightarrow 0$, when $n \to +\infty$, the series is therefore divergent and consequently: $D_1 = \left]-\dfrac{1}{a}, \dfrac{1}{a}\right[$.

The sum:

$$\begin{aligned} S_1 &= f_1(x) = \sum_{n=0}^{+\infty} a^n x^n \\ &= \frac{1}{1-ax}, \text{ for all } x \in \left]-\frac{1}{a}, \frac{1}{a}\right[. \end{aligned}$$

2. $f_2(x) = \sum\limits_{n=0}^{+\infty} (n+1)x^n$.

Radius of convergence:

We have $a_n = n + 1$, which implies that:

$$R = \left[\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|\right]^{-1} = 1.$$

Domain of convergence: if $|x| = 1$, we have

$$|(n+1)x^n| = n + 1 \to +\infty \neq 0, \text{ when } n \to +\infty,$$

the series is therefore divergent and consequently: $D_2 =]-1, 1[$.

Solution of Exercise 1

The sum:

$$\begin{aligned} S_2 &= f_2(x) = \sum_{n=0}^{+\infty}(n+1)x^n \\ &= \sum_{n=1}^{+\infty} nx^n + \sum_{n=0}^{+\infty} x^n \\ &= x\sum_{n=1}^{+\infty} nx^{n-1} + \sum_{n=0}^{+\infty} x^n \\ &= \frac{x}{(1-x)^2} + \frac{1}{1-x}, \text{ for all } x \in]-1,1[\end{aligned}$$

3. $f_3(x) = \sum_{n=0}^{+\infty} n^2x^n$.
Radius of convergence:

We have $a_n = n^2$, which implies that:

$$R = \left[\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|\right]^{-1} = 1.$$

Domain of convergence: If $|x| = 1$, we have:

$$|(n+1)x^n| = n^2 \to +\infty \neq 0, \text{ when } n \to +\infty,$$

the series is therefore divergent and consequently: $D_3 =]-1,1[$.

The sum

$$\begin{aligned} S_3 &= f_3(x) = \sum_{n=0}^{+\infty} n^2x^n \\ &= \sum_{n=1}^{+\infty} nx^n + \sum_{n=2}^{+\infty} n(n-1)x^n \\ &= x\sum_{n=1}^{+\infty} nx^{n-1} + x^2\sum_{n=2}^{+\infty} n(n-1)x^{n-2} \\ &= \frac{x}{(1-x)^2} + \frac{2x}{(1-x)^3}, \text{ for all } x \in]-1,1[\end{aligned}$$

4. $f_4(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{2^n}{n+1}x^n$.
Radius of convergence:

We have $a_n = (-1)^n\frac{2^n}{n+1}$, which implies that: $R = \left[\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|\right]^{-1} = \frac{1}{2}$.

Solution of Exercise 1

Domain of convergence: If $x = \frac{1}{2}$, we have

$$(-1)^n \frac{2^n}{n+1} x^n = \frac{(-1)^n}{n+1}$$

general term of Leibniz series, which is convergent.

If $x = -\frac{1}{2}$, we have:

$$(-1)^n \frac{2^n}{n+1} x^n = \frac{1}{n+1} \overset{V(+\infty)}{\sim} \frac{1}{n},$$

general term of harmonic series, which is divergent.

Finally $D_4 = \left]-\frac{1}{2}, \frac{1}{2}\right]$.

The sum:

$$\begin{aligned} S_4 &= f_4(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{2^n}{n+1} x^n \\ &= \frac{1}{x} \sum_{n=0}^{+\infty} (-1)^n \frac{2^n}{n+1} x^{n+1} \\ &= \frac{1}{2x} \ln(1+2x), \text{ for all } x \in \left]-\frac{1}{2}, \frac{1}{2}\right]. \end{aligned}$$

Solution of Exercise 2

We consider a power series with general term $a_n t^n$, radius of convergence $R > 0$, and sum S. We assume that S is a solution of the differential equation:

$$(1 + t^2) \ddot{f}(t) = 2f(t). \tag{1}$$

1. Relationship between the coefficients a_n and a_{n+2}:
Let $f(t) = \sum_{n=0}^{+\infty} a_n t^n$. The equation (1) becomes:

$$\sum_{n=2}^{+\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{+\infty} n(n-1) a_n t^n = 2 \sum_{n=0}^{+\infty} a_n t^n,$$

or in an equivalent manner:

$$\sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} t^n + \sum_{n=0}^{+\infty} n(n-1) a_n t^n = 2 \sum_{n=0}^{+\infty} a_n t^n.$$

By grouping terms of the same degree, we obtain $a_{n+2} = -\frac{(n-2)}{n+2} a_n$.
2. By taking $n = 2$, we find $a_4 = 0$, therefore also $a_{2p} = 0$, for all $p > 2$.
3. The conditions $s(0) = 0$ and $\dot{s}(0) = 1$ imply that $a_0 = 0$ and $a_1 = 1$, therefore $a_2 = 0$.
The value of a_{2p+1}, for $p \in \mathbb{N}$ is calculated recursively from a_1 :
For $n = 1$, $a_3 = \dfrac{1}{3}$ and for $n = 3$, $a_5 = -\dfrac{1}{5} a_3 = -\dfrac{1}{3 \times 5}$.
Suppose that the proposition in question is true up to the order $2p - 1$, $p \geq 2$; i.e

$$a_{2p-1} = \frac{(-1)^p}{(2p-1)(2p-3)},$$

we then have

$$\begin{aligned} a_{2p+1} &= -\frac{2p-3}{2p+1} \times \frac{(-1)^p}{(2p-1)(2p-3)} \\ &= \frac{(-1)^{p+1}}{(2p-1)(2p+1)}. \end{aligned}$$

The formula is verified in the order $2p + 1$ and therefore the recurrence is indeed verified and we have:

$$S(t) = \sum_{p=0}^{+\infty} \frac{(-1)^{p+1} t^{2p+1}}{(2p-1)(2p+1)}.$$

Solution of Exercise 2

4.The power series of general term $a_n t^n$ is bounded above on $[-1,+1]$ by the numerical series of general term $\dfrac{1}{(n-1)(n+1)}$, which is a convergent series. Therefore the power series of general term $a_n t^n$ converges normally on $[-1,+1]$.
Its radius of convergence is 1 because $\lim_{n\to+\infty} \dfrac{a_{n+2}}{a_n} = 1$.

Hence, by differentiating term by term on $[-1,+1]$, we obtain:

$$\dot{g}(t) = \sum_{p=1}^{+\infty}(-1)^{p+1}t^{2p-2}\sum_{p=0}^{+\infty}(-1)^{p}t^{2p} = \frac{1}{1+t^2}.$$

By integrating, with the condition $g(0) = 0$, we find:

$$g(t) = \arctan(t).$$

6.We can deduce that the function S can be written as:

$$S(t) = \int_0^t t\arctan(t)dt + t.$$

Integrating by parts, we find:

$$S(t) = \frac{1}{2}\left(t^2+1\right)\arctan(t) + \frac{t}{2}.$$

Solution of Exercise 3

Consider the function f defined by:

$$f(x,y) = \frac{1 - x\cos(y)}{x^2 - 2x\cos(y) + 1}.$$

1. Let's devlop f as a power series according to the powers of x.

We have:

$$\begin{aligned} f(x,y) &= \frac{1 - x\cos(y)}{x^2 - 2x\cos(y) + 1} \\ &= \frac{1}{2}\left(\frac{1}{1 - x\exp(iy)} + \frac{1}{1 - x\exp(-iy)}\right). \end{aligned}$$

Hence, for $|x| < 1$, we have:

$$\begin{aligned} f(x,y) &= \frac{1}{2}\sum_{n=0}^{+\infty} x^n \left(\exp(iny) + \exp(-iny)\right) \\ &= \sum_{n=0}^{+\infty} x^n \cos(ny). \end{aligned}$$

2. Let $|x| < 1$ be fixed. The series of functions with general term $x^n cos(ny)$ is normally convergent on the domain $y \in [0, \frac{\pi}{2}]$ because it is dominated by the convergent numerical series with general term x^n. We can therefore integrate it term by term on $[0, \frac{\pi}{2}]$; that is:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} f(x,y)dy &= \sum_{n=0}^{+\infty} \int_0^{\frac{\pi}{2}} x^n \cos(ny)dy \\ &= \int_0^{\frac{\pi}{2}} dy + \sum_{n=1}^{+\infty} \int_0^{\frac{\pi}{2}} x^n \cos(ny)dy \\ &= \frac{\pi}{2} + \sum_{n=1}^{+\infty} \left[\frac{\sin(ny)}{n}\right]_0^{\frac{\pi}{2}} x^n \\ &= \frac{\pi}{2} + \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \frac{\pi}{2} + g(x). \end{aligned}$$

On the other hand

$$\dot{g}(x) = \sum_{n=0}^{+\infty} (-1)^n x^{2n} = \sum_{n=0}^{+\infty} (-x^2)^n = \frac{1}{1+x^2}, \quad |x| < 1.$$

Therefore $\int_0^{\frac{\pi}{2}} f(x,y)dy = \frac{\pi}{2} + \arctan(x), \ \ |x| < 1.$