Abdelhafid Boussouf University of Mila — Academic year 2025-2026
Faculty of mathematics and computer science — Department of Mathematics
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Series of Exercises 3

Exercise 1

Determine the radius and the domain of convergence of the following power series, and if possible, compute their sums:

1. $f_1(x) = \sum_{n=0}^{+\infty} a^n x^n \ (a \neq 0)$
2. $f_2(x) = \sum_{n=0}^{+\infty} (n+1)x^n$
3. $f_3(x) = \sum_{n=0}^{+\infty} n^2 x^n$
4. $f_4(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{2^n}{n+1} x^n$.

Exercise 2

Consider a power series with general term $a_n t^n$, with radius of convergence $R > 0$, and sum S. Assume that S is a solution of the differential equation.

$$(1+t^2)\ddot{f}(t) = 2f(t). \tag{1}$$

❶ Establish a relationship linking, for each $n \in \mathbb{N}$, the coefficients a_n and a_{n+2}.

❷ Determine the value of a_4 and then a_{2p}, for all $p > 2$.

❸ We now assume that $S(0) = 0$ and $\dot{S}(0) = 1$. Calculate a_0, a_2 and the value of a_{2p+1}, for all $p \in \mathbb{N}$.

❹ Show that the power series with general term $a_n t^n$ converges normally on the interval $[-1, +1]$. What is its radius of convergence?

❺ Let $g(0) = 0$ and $g(t) = \frac{\dot{S}(t) - 1}{t}$, for $t \neq 0$.
Calculate the derivative $\dot{g}$ of g (you will find a simple rational fraction).

❻ Deduce from answer 5, an explicit solution to equation (1).

Exercise 3

Consider the function f defined by:

$$f(x,y) = \frac{1 - x\cos(y)}{1 + x^2 - 2x\cos(y)}.$$

❶ Find an interval $]-R, R[$, such that for all $t \in \mathbb{R}$, the function f can be developed as a power series according to the powers of $x \in]-R, R[$, and give this development.

❷ Assume that $x \in]-1, 1[$. Calculate $\int\limits_0^{\frac{\pi}{2}} f(x,y)dy$.

Exercise 4 (Additional)

Find an interval $]-R, R[$, such that each of the following functions can be developed on $]-R, R[$ as a power series and give this development:

❶ $f_1(x) = \dfrac{1}{x-2}$.

❷ $f_2(x) = \dfrac{1}{(x-2)^2}$.

❸ $f_3(x) = \dfrac{2x}{(1+x)(2+x)}$.

❹ $f_4(x) = -\dfrac{x}{2} + x\cos^2(x) + \dfrac{-1}{2}\sin(2x)$.

❺ $f_5(x) = \int_0^x \dfrac{\sin t}{t} dt$.