Statistical Tests for a Mean and a Proportion

Course Notes with Detailed Examples

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1 Introduction

In statistical inference, we often want to know whether the value observed in a sample is compatible with a **theoretical value** announced for the population.

Typical questions:

- Is the average weight of a product really equal to the nominal value?
- Is the proportion of defective items lower than a certain threshold?

To answer such questions, we use **hypothesis tests**.

We always formulate two hypotheses:

- **Null hypothesis ($\mathbf{H}_0$)**: the parameter (mean, proportion, etc.) is equal to the theoretical value.
- **Alternative hypothesis ($\mathbf{H}_1$)**: the parameter is different from, greater than, or less than this theoretical value.

The test uses the sample to decide whether we have enough evidence to **reject** the null hypothesis H_0 in favor of H_1.

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2 Test on a Mean

We observe a sample of size n from a population with (unknown) mean μ. The sample has mean $\bar{X}$.

We want to test:

$$H_0 : \mu = \mu_0 \quad \text{against} \quad H_1 : \mu \neq \mu_0$$

(or sometimes $H_1 : \mu > \mu_0$ or $H_1 : \mu < \mu_0$ for one-sided tests).

There are two main cases:

1. Population variance σ^2 is known (theoretical case).
2. Population variance σ^2 is unknown (practical case).

2.1 Case 1: Population Variance Known

Suppose the population variance σ^2 is known. Then the test statistic is:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Under H_0, if the sample comes from a normal population or n is large, we have:

$$Z \sim \mathcal{N}(0, 1)$$

We then compare the observed value z_{obs} of Z to the critical values of the standard normal distribution.

2.2 Case 2: Population Variance Unknown

In practice, the population variance σ^2 is usually unknown. We estimate it using the sample standard deviation:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{X})^2}$$

The test statistic becomes:

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

Under H_0, if the population is normal, this statistic follows a Student distribution with $n-1$ degrees of freedom:

$$t \sim t_{n-1}$$

We then compare the observed value t_{obs} to the critical values of the t distribution with $n-1$ degrees of freedom.

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3 Test on a Proportion

We now consider a binary variable (success/failure, defective/non-defective, etc.). Let p be the **true proportion** in the population.

We observe a sample of size n, containing X successes. The empirical (sample) proportion is:

$$\hat{p} = \frac{X}{n}$$

We want to test:

$$H_0 : p = p_0 \quad \text{against} \quad H_1 : p \neq p_0 \quad (\text{or } p > p_0 \text{ or } p < p_0)$$

Under H_0, if n is large enough, $\hat{p}$ can be approximated by a normal distribution:

$$\hat{p} \approx \mathcal{N}\left(p_0, \frac{p_0(1-p_0)}{n}\right)$$

The test statistic is thus:

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Under H_0,

$$Z \sim \mathcal{N}(0, 1)$$

We compare the observed z_{obs} with the critical values of the standard normal distribution.

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4 Numerical Examples (Detailed)

In this section, we present several detailed examples for tests on a mean and on a proportion.

4.1 Example 1: Test on a Mean (Unknown Variance, Two-Sided)

A professor claims that the average score on an exam is 12 out of 20. A student takes a sample of $n = 16$ exam papers and finds:

- sample mean: $\bar{X} = 11.3$,
- sample standard deviation: $s = 2.4$.

We test, at the 5% significance level:

$$H_0 : \mu = 12 \quad \text{against} \quad H_1 : \mu \neq 12$$

Step 1: Test Statistic

The population variance is unknown, so we use the t-test:

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{11.3 - 12}{2.4/\sqrt{16}} = \frac{-0.7}{2.4/4} = \frac{-0.7}{0.6} \approx -1.167$$

Step 2: Critical Value

Degrees of freedom: $n - 1 = 15$.

For a two-sided test at significance level $\alpha = 0.05$, the critical value is:

$$t_{0.025,15} \approx 2.13$$

The rejection region is:

$$|t| > 2.13$$

Step 3: Decision

We have:

$$|t_{\text{obs}}| \approx 1.167 < 2.13$$

So the test statistic does not fall into the critical region.

Conclusion: We do **not** reject H_0 at the 5% level. The data are compatible with the claim that the true mean is 12.

4.2 Example 2: Test on a Mean (Known Variance, Two-Sided)

A company produces packages that are supposed to weigh 500 grams on average. From past data, the population standard deviation is known to be $\sigma = 10$ grams.

A sample of $n = 100$ packages is taken. The sample mean is:

$$\bar{X} = 505 \text{ grams}$$

We test:

$$H_0 : \mu = 500 \quad \text{against} \quad H_1 : \mu \neq 500$$

Step 1: Test Statistic

Since σ is known, we use the Z-test:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{505 - 500}{10/\sqrt{100}} = \frac{5}{10/10} = \frac{5}{1} = 5$$

Step 2: Critical Value

For a two-sided test at $\alpha = 0.05$:

$$z_{0.025} \approx 1.96$$

The rejection region is $|Z| > 1.96$.

Step 3: Decision

We observe:

$$|Z_{\text{obs}}| = 5 > 1.96$$

Conclusion: We **reject** H_0 at the 5% level. The mean weight is significantly different from 500 grams.

4.3 Example 3: Test on a Proportion (Two-Sided)

A manufacturer claims that 90% of its products are non-defective. In a quality control procedure, a sample of $n = 200$ products is tested. Among them, 170 are non-defective.

The empirical proportion is:

$$\hat{p} = \frac{170}{200} = 0.85$$

We test, at the 5% significance level:

$$H_0 : p = 0.90 \quad \text{against} \quad H_1 : p \neq 0.90$$

Step 1: Test Statistic

Under H_0, the standard error is:

$$\sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.9 \times 0.1}{200}} = \sqrt{\frac{0.09}{200}}$$

Compute:

$$\frac{0.09}{200} = 0.00045 \quad \Rightarrow \quad \sqrt{0.00045} \approx 0.0212$$

Then:

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.85 - 0.90}{0.0212} = \frac{-0.05}{0.0212} \approx -2.36$$

Step 2: Critical Value

For a two-sided test at $\alpha = 0.05$:

$$z_{0.025} \approx 1.96$$

Rejection region: $|Z| > 1.96$.

Step 3: Decision

$$|Z_{\text{obs}}| \approx 2.36 > 1.96$$

Conclusion: We **reject** H_0. The data suggest that the true proportion of non-defective products is significantly different from 90%, and since $\hat{p} = 0.85 < 0.90$, it appears to be lower.
