

Chapter 4

Predicate Logic

Introduction

Consider the famous syllogism:

Every man is mortal
Socrates is a man
Therefore, Socrates is mortal

With propositional logic, we cannot express this syllogism precisely.

The language of predicates has the elements and tools that allow us to represent this kind of statement.

Language of Predicates

1. Alphabet:

- Logical connectives: \neg , \wedge , \vee , \rightarrow , \leftrightarrow

- Quantifiers: \forall , \exists

- Variables: x , y , \dots

- Constants: a , b , \dots

- Predicate symbols: P , Q , \dots

- Function symbols: f , g , \dots

2. Functions

This is a generalization of numerical functions over any domain D (people, cities, ...).

Examples:

- $D = \mathbb{R}$: $f(x) = x^2$

- $D = \text{set of humans}$: $g(x) = \text{father}(x)$

- $D = \mathbb{N}$: $h(x, y) = \text{gcd}(x, y)$

The result of the function is an element of the same domain.

3. Predicates

This is a property of an element of the domain or a relation between elements of the domain.

Examples:

- $P(x)$: x is a prime number - $Q(x, y)$: $x > y$

The value of the predicate is Boolean: True or False.

4. Quantifiers

- Universal quantifier \forall (for all)

- Existential quantifier \exists (there exists at least one)

Example:

If we have: $P(x)$: x is present, $A(x)$: x is absent

$\forall xP(x)$: All students are present

$\exists xA(x)$: There exist students who are absent

5. Terms

- Every constant is a term

- Every variable is a term

- If t_1, \dots, t_n are terms and f is a function then:

$f(t_1, \dots, t_n)$ is a term.

6. Formulas

- If t_1, \dots, t_n are terms and P is a predicate then:

$P(t_1, \dots, t_n)$ is a formula. - If x is a variable and α, β are two formulas then:

$\neg\alpha, \alpha \wedge \beta, \alpha \vee \beta, \alpha \rightarrow \beta, \alpha \leftrightarrow \beta, \forall x\alpha, \exists x\alpha$ are formulas.

Examples of formulas

$\forall xP(x) \wedge \exists yQ(y) \rightarrow P(x),$

$\forall x\forall y\exists z(P(x, y) \rightarrow Q(x, y, z))$

Operator Precedence + Quantifiers:

$\neg, \wedge, \vee, (\forall, \exists), \rightarrow, \leftrightarrow$

Scope of a Quantifier

The scope of a quantifier in a formula is the subformula concerned by that quantifier.

Example: $\forall x(P(x) \wedge \exists yQ(y)) \rightarrow P(x)$

- The scope of \exists is $Q(y)$
- The scope of \forall is $P(x) \wedge \exists yQ(y)$

Free and Bound Variables

- An occurrence of x in a formula α is bound if it appears within the scope of a quantifier, otherwise it is free.

- A variable is bound if it has at least one bound occurrence.
- A variable is free if it has at least one free occurrence.

Example:

- $P(x) \rightarrow Q(y, z)$: all variables are free
- $\forall xP(x) \wedge Q(y)$: x is bound and y is free

Closed Formula

This is a formula where all variables are bound and not free.

Example:

$$\forall x \forall y \exists z (P(x, y) \rightarrow Q(x, y, z))$$

Interpretation

To assign a truth value to a formula, we must give a meaning to all symbols of the formula (predicates, functions, constants) as well as the domain of the variables.

Example 1:

$$\alpha = \forall x \exists y P(x, y), D = \mathbb{N}$$

- If P means $<$ then α is true
- If P means $>$ then α is false

Evaluation

The evaluation of a formula consists of giving a value to each free variable of the formula.

Assigning a truth value to a formula requires an interpretation and an evaluation of the formula.

Example:

$$\beta = P(x, f(x))$$

Let interpretation I be such that $I(P)$: “=” and $I(f)$: square of ...

- If $x = 2$ then β is false

- If $x = 1$ then β is true

Satisfiability

A formula α is satisfiable if there exists an interpretation I and an evaluation v for which α is true. We say that the evaluation v satisfies α for the interpretation I , and we write $I \models \alpha[v]$.

Example: $\alpha = P(f(x, y), y)$, such that:

$D = \mathbb{N}$, $I(P) = “>”$, $I(f) = “-”$, $v(x) = 4$, $v(y) = 1$

$$\begin{aligned} I(\alpha)[v] &= I(P(f(x, y), y)) \\ &= I(P)(I(f(x, y)), v(y)) \\ &= I(P)(I(f)(v(x), v(y)), v(y)) \\ &=> (-4, 1, 1) => (3, 1) \end{aligned}$$

A set of formulas Γ is satisfiable if there exists an interpretation I and an evaluation v for which all formulas of Γ are true.

Validity of a Formula

A formula α is valid if it is true for every interpretation I and every evaluation v . We denote $\models \alpha$.

Example: $\beta = \neg P(x) \vee P(x)$.

To show that a formula is not valid, it suffices to find an interpretation and/or an evaluation for which the formula is false.

Logical Consequence

We say that β is a logical consequence of Γ (denoted $\Gamma \models \beta$) if and only if for every interpretation I and every evaluation v , we have:

If $I \models \Gamma[v]$ then $I \models \beta[v]$

Example: $\forall x(\alpha \rightarrow \beta)$, $\forall x\alpha \models \forall x\beta$

Normal Forms

1- Prenex Normal Form

We say that α is in prenex form if:

- α is of the form: $Q_1x_1 \dots Q_nx_n\beta$ such that: $Q_i \in \{\forall, \exists\}$
- The scope of Q_nx_n is β
- β contains no quantifiers.

To transform formulas into prenex normal form, we use the following properties:

$$\begin{aligned} \forall x\alpha \rightarrow \beta &\equiv \exists x(\alpha \rightarrow \beta) && \text{(Condition: } x \text{ does not appear free in } \beta) \\ \exists x\alpha \rightarrow \beta &\equiv \forall x(\alpha \rightarrow \beta) \\ \alpha \rightarrow \forall x\beta &\equiv \forall x(\alpha \rightarrow \beta) && \text{(Condition: } x \text{ does not appear free in } \alpha) \\ \alpha \rightarrow \exists x\beta &\equiv \exists x(\alpha \rightarrow \beta) \end{aligned}$$

Example: $\exists xP(x) \rightarrow \exists yP(y) \dots (1)$

x does not appear free in $\exists yP(y)$

$$(1) \equiv \forall x(P(x) \rightarrow \exists yP(y))$$

y does not appear free in $P(x)$

$$(1) \equiv \forall x\exists y(P(x) \rightarrow P(y))$$

Remark:

If the conditions for the properties are not satisfied, we proceed with modifications according to the following rules:

- $\forall x\beta = \forall y\beta[y/x]$
- $\exists x\beta = \exists y\beta[y/x]$

We obtain $\beta[y/x]$ by replacing x with y .

Example:

Transform into prenex normal form:

$$\exists xP(x, y) \rightarrow \exists yQ(y, x)$$

- x appears free in “ $\exists yQ(y, x)$ ”, we change the variable
- $\exists uP(u, y) \rightarrow \exists yQ(y, x)$

- u does not appear free in " $\exists yQ(y, x)$ "
- $\forall u(P(u, y) \rightarrow \exists yQ(y, x))$
- y appears free in " $P(u, y)$ ", we change the variable
- $\forall u(P(u, y) \rightarrow \exists vQ(v, x))$
- u does not appear free in " $P(u, y)$ "
- $\forall u\exists v(P(u, y) \rightarrow Q(v, x))$

2- Skolem Normal Form

This consists of eliminating all existential quantifiers by using new function symbols or constants while preserving the satisfiability of the formula.

Given: $\alpha = \forall x\exists yP(x, y)$ this formula means that for each x there exists a y that satisfies $P(x, y)$, i.e., we can define a function $f(x)$ that will replace y in the formula.

The Skolem form of α is therefore: $\alpha_s = \forall xP(x, f(x))$

Example 1:

$$\beta = \forall x_1\forall x_2\exists yP(x_1, x_2, y)$$

$$\beta_s = \forall x_1\forall x_2P(x_1, x_2, f(x_1, x_2))$$

For a formula of the form $\exists x\beta$, we eliminate the quantifier and replace x with a new constant.

Example 2:

$$\alpha = \exists xP(x) \Rightarrow \alpha_s = P(a)$$

Remarks:

- Skolemization of a formula assumes it is in prenex form.
- Skolemization of a formula does not yield an equivalent formula.

Example 3:

$$\alpha = \exists x\forall y\forall z\exists vP(x, y, z, v)$$

$$\alpha_s = \forall y\forall zP(a, y, z, f(y, z))$$