Chapter 4

Statistical Tests

Let H_0 be a hypothesis concerning a population. Based on the results of samples taken from this population, we are led to accept or reject the hypothesis H_0. The decision rules are called statistical tests. H_0 denotes the so-called null hypothesis and H_1 denotes the so-called alternative hypothesis.

We may have H_0 true and H_1 false, or H_0 false and H_1 true.

Homogeneity Tests

From a sample of size n_1 taken from a population P_1 and a sample of size n_2 taken from a population P_2, the test allows us to decide :

$$\begin{cases} H_0 : \theta_0 = \theta_1 \\ H_1 : \theta_0 \neq \theta_1 \end{cases}$$

where θ_0 and θ_1 are the two values of the same parameter of the two populations P_1 and P_2.

4.1 Student's t-test (comparison of two means)

Let X and Y be two independent random variables with means m_1 and m_2 and standard deviations σ_1 and σ_2. We have two independent samples $\{X_1; X_2; \ldots; X_{n_1}\}$ such that X_i follows the same distribution $N(m_1, \sigma_1)$ and $\{Y_1; Y_2; \ldots; Y_{n_2}\}$ such that Y_i follows the same distribution $N(m_2, \sigma_2)$. We want to determine whether the means m_1 and m_2 are significantly different or not; we use Student's t-test :

a– If $n_1 \geq 30, n_2 \geq 30$ and σ_1, σ_2 are known.

We test at the significance level α :

$$\begin{cases} H_0 : m_1 = m_2 \\ H_1 : m_1 \neq m_2 \end{cases}$$

– We accept H_0 (i.e., there is no significant difference between the means of the two samples) if

$$z \in]-u; u[$$

where $z = \frac{\bar{x}-\bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$ and the value u is read from the standard normal table $N(0, 1)$ such that $\Phi(u) = 1 - \frac{\alpha}{2}$.

– We reject H_0 if $z \notin]-u; u[$ (there is a significant difference).

Remark 1 If σ_1 and σ_2 are unknown, we replace them by the estimators :

$$\hat{S}_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2, \qquad \hat{S}_2^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} (y_j - \bar{y})^2$$

$$\text{that is, } z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\hat{S}_1^2}{n_1} + \frac{\hat{S}_2^2}{n_2}}}$$

b– If $n_1 < 30, n_2 < 30$ and σ_1, σ_2 are equal and unknown $(\sigma_1 = \sigma_2 = \sigma)$

– We accept H_0 (i.e., there is no significant difference between the means of the two samples) if

$$z \in]-t_{n_1+n_2-2,\frac{\alpha}{2}}; t_{n_1+n_2-2,\frac{\alpha}{2}}[$$

where

$$z = \frac{\bar{x} - \bar{y}}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

with

$$S = \sqrt{\frac{(n_1 - 1)\,\hat{S}_1^2 + (n_2 - 1)\,\hat{S}_2^2}{n_1 + n_2 - 2}}$$

and the value $t_{n_1+n_2-2,\frac{\alpha}{2}}$ is read from Student's t-table with $k = n_1 + n_2 - 2$ degrees of freedom (d.f.) and $\gamma = \frac{\alpha}{2}$.

– We reject H_0 if $z \notin]-t_{n_1+n_2-2,\frac{\alpha}{2}}; t_{n_1+n_2-2,\frac{\alpha}{2}}[$ (there is a significant difference).

4.2 Comparison of Two Proportions

Let two populations P_1 and P_2. We extract a sample of size n_1 from population P_1, and a sample of size n_2 from population P_2.

We compare two unknown proportions p_1 and p_2. We want to test whether they are the same. The null hypothesis is

$$H_0 : \ll p_1 = p_2 \gg \quad \text{against} \quad H_1 : \ll p_1 \neq p_2 \gg$$

We have two series of observations, of size n_1 for p_1 estimated by f_1, and of size n_2 for p_2 estimated by f_2.

– We accept H_0 (i.e., equality of proportions) if

$$z \in]-u; u[$$

where

$$z = \frac{f_1 - f_2}{\sqrt{f(1-f)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

with

$$f = \frac{n_1 f_1 + n_2 f_2}{n_1 + n_2}$$

and u is read from the standard normal table $N(0,1)$ such that $\Phi(u) = 1 - \frac{\alpha}{2}$.
– We reject H_0 if $z \notin] - u; u[$ (there is a significant difference between the proportions of the two samples).

4.3 Fisher's Test (Comparison of Two Variances)

Let X and Y be two independent random variables with means m_1 and m_2 and standard deviations σ_1 and σ_2. We have two independent samples $\{X_1; X_2; \ldots; X_{n_1}\}$ where X_i follows $N(m_1, \sigma_1)$ and $\{Y_1; Y_2; \ldots; Y_{n_2}\}$ where Y_i follows $N(m_2, \sigma_2)$. We want to determine whether the variances σ_1^2 and σ_2^2 are significantly different or not, using Fisher's test.

We set the hypothesis $H_0 : \sigma_1 = \sigma_2$ (the two populations have the same variance) and

$$F = \begin{cases} \frac{\hat{S}_1^2}{\hat{S}_2^2} & \text{if } \hat{S}_1^2 > \hat{S}_2^2 \\ \frac{\hat{S}_2^2}{\hat{S}_1^2} & \text{if } \hat{S}_1^2 < \hat{S}_2^2 \end{cases}$$

where

$$\hat{S}_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2, \quad \hat{S}_2^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} (y_j - \bar{y})^2$$

If $F < F^{\alpha}_{n_1-1,n_2-1}$, we accept H_0. If $F > F^{\alpha}_{n_1-1,n_2-1}$, we reject H_0 ; there is a significant difference between the variances of the two samples. $F^{\alpha}_{n_1-1,n_2-1}$ is read from Fisher's table at significance level α with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

4.4 Chi-Square Tests

We can distinguish three types of Chi-square (χ^2) tests :

- Chi-square goodness-of-fit test (H_0 : "Does the character X follow a specific distribution ?"),
- Chi-square homogeneity test (H_0 : "Does the character X follow the same distribution in two given populations ?"),
- Chi-square independence test (H_0 : "Are the characters X and Y independent ?").

These three tests have a common principle : we divide observations into k classes, with observed counts $n_1 = N_1(w), \ldots, n_k = N_k(w)$. The hypothesis H_0 allows us to compute theoretical counts $n_{1,th}, \ldots, n_{k,th}$. We reject H_0 if the observed counts differ too much from the theoretical counts.

We accept H_0 if

$$h \notin]\chi_{k-1-m,\alpha}; +\infty[$$

where

$$h = \sum_{i=1}^{k} \frac{(n_i - n_{i,th})^2}{n_{i,th}}$$

Here, $\chi_{k-1-m,\alpha}$ is read from the Chi-square table with $(k-1-m)$ degrees of freedom (df), $\gamma = \alpha$, k is the number of classes, and m is the number of parameters estimated to compute theoretical counts.

We reject H_0 if

$$h \in]\chi_{k-1-m,\alpha}; +\infty[$$