Chapter 4: Forced Linear Systems with One Degree of Freedom

Introduction

Forced vibrations (oscillations) occur when a system, during its motion, is subjected to an external periodic force. Such forces are often referred to as *external excitations*. The resulting motion is called the *response* of the system to the external excitation.

The excitation force may be harmonic, non-harmonic periodic, non-periodic, or random. In this course, we will focus only on harmonic excitations.

A harmonic excitation can be expressed mathematically as:

$f(t) = f_0 \sin(\omega t + \varphi)$, $f(t) = f_0 \cos(\omega t + \varphi)$, $f(t) = f_0\, e^{i(\omega t + \varphi)}$

where :

φ : the phase of the excitation (depends on the value of the force f at t = 0

Lagrange's equation for forced systems

If, in addition to the friction $f = -\alpha \dot{q}$, there exists an external excitation force F(t), the Lagrange equation can be written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F_{ext}$$

Equation of motion for forced systems

The equation of motion for a forced system in the presence of a damping force is defined as follows:
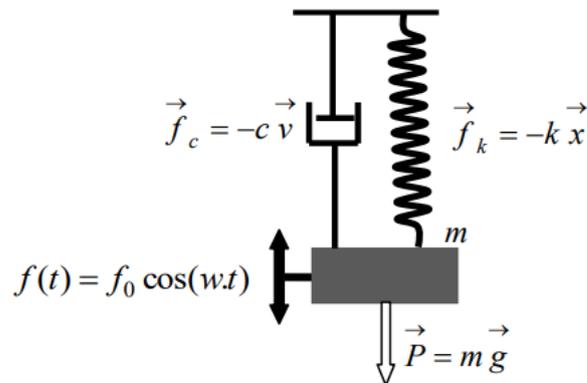
$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = F(t)$$

where F(t) is called the external excitation function.

This equation is a second-order linear non-homogeneous differential equation with constant coefficients.

Example:

If a sinusoidal external force is applied to the damped mass-spring system:

$f(t) = f_0 \cos(\omega t)$

- Lagrangian L=T-U
- Kinetic energy $T=\frac{1}{2}mv^2$
- Potential energy $U=\frac{1}{2}Kx^2$

$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}Kx^2$$

- The dissipation fonction is $D=\frac{1}{2}\alpha\dot{x}^2$
- **Lagrange's equation with dissipation**

  The generalized Lagrange equation including damping is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}+\frac{\partial D}{\partial \dot{q}}=F_{ext}$$

After derivation, we obtain:

$m\ddot{x}+kx+\alpha\dot{x}=f_0\cos(\omega t)$

$$\ddot{x}+\frac{\alpha}{m}\dot{x}+\frac{k}{m}x=\frac{f_0\cos(\omega t)}{m}$$

This is the equation of motion **of a** damped, forced harmonic oscillator

Solution of the differential equation

The equation of motion is a second-order differential equation with a nonzero right-hand side:

mẍ+αẋ+Kx=F(t)

This equation's general solution is the sum of two parts**:**

x(t)=xh(t)+xp(t)

where:

- $x_h(t)$ is the **homogeneous solution** (the solution of the corresponding equation without the external force, i.e. F(t)=0);
- $x_p(t)$ is the **particular solution** (the response due to the external excitation F(t)).

❖ **General solution xh(t)** of the homogeneous equation (without a forcing term). It is called transient because it dies out over time.

❖ **Particular solution xp(t)** of the non-homogeneous equation (with a forcing term). It is called steady-state because it persists throughout the motion.

The solution of the homogeneous equation

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

depends on the nature of the roots of the characteristic equation

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$$

Therefore, we focus only on the particular solution, which has the form

$$x(t) = x_p(t) = X\cos(\omega t + \varphi)$$

To find the constants X and ϕ, we differentiate xp(t) and substitute it back into the equation.

$$\dot{x}_p = -Xw\sin(w.t + \varphi) \text{ et } \ddot{x}_p = -Xw^2\cos(w.t + \varphi)$$

$$X\left[\left(k - mw^2\right)\cos(w.t + \varphi) - cw\sin(w.t + \varphi)\right] = f_0\cos(w.t)$$

We use :

$$\cos(w.t + \varphi) = \cos(wt)\cos\varphi - \sin\varphi\sin wt$$

$$\sin(w.t + \varphi) = \cos\varphi\sin(w.t) + \sin\varphi\cos(w.t)$$

we find

$$\left(k - mw^2\right)\cos\varphi - cw\sin\varphi = \frac{f_0}{x}$$

$$\left(k - mw^2\right)\sin\varphi + cw\cos\varphi = 0$$

$$X = \frac{f_0}{\sqrt{\left(k - mw^2\right)^2 + c^2 w^2}}$$

$$\varphi = \tan^{-1}\left(\frac{-cw}{k - mw2}\right)$$

We have ; $\omega_0 = \sqrt{\dfrac{k}{m}}$ et $\quad \varepsilon = \dfrac{c}{Cc} = \dfrac{C}{2\sqrt{mk}}$ therefore, we can write : $\dfrac{c2w2}{K2} = \left(2\,\varepsilon\,\dfrac{W}{W_0}\right)^2$

The system's response to the excitation force can be written in the form:

$$X = \frac{\dfrac{f_0}{k}}{\sqrt{\left(1 - \left(\dfrac{w}{w_n}\right)^2\right)^2 + 4\varepsilon^2\left(\dfrac{w}{w_n}\right)^2}}$$

$$\varphi = \tan^{-1}\left(-\frac{2\varepsilon\left(\dfrac{w}{w_n}\right)}{1 - \left(\dfrac{w}{w_n}\right)^2}\right)$$

Excitation of an undamped system ($\varepsilon = 0$).

Lagrange's equation of motion:

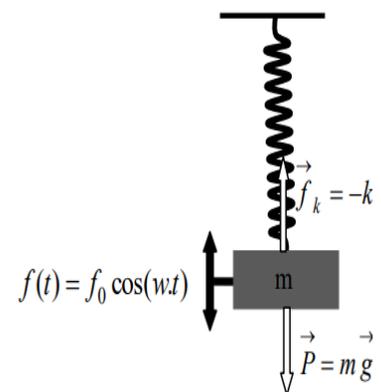$$\ddot{x} + \frac{k}{m}x = \frac{f_0 \cos(\omega t)}{m}$$

Solution of the differential equation:

$$x(t) = x_H(t) + x_p(t)$$

The solution of the homogeneous equation: $\ddot{x} + \dfrac{k}{m}x = 0$

Is given by: $x_H(t) = A\cos(\omega t + \varphi)$

$\vec{f}_k = -k$

$f(t) = f_0 \cos(wt)$ m

$\vec{P} = m\vec{g}$

A and φ are constants that can be determined from the initial conditions.

The particular solution is given by:

$x_p(t) = X\cos(\omega t + \varphi)$

where X is the amplitude and φ is the phase shift of the total solution.

$$X = \frac{\delta_{st}}{1 - \left(\dfrac{w}{w_n}\right)^2}$$

and $\varphi = 0$

thus ;

$$x_p = \frac{\delta_{st}}{1 - \left(\dfrac{w}{w_n}\right)^2} \cos(w.t)$$

Amplification factor : As we saw earlier, the amplification factor is defined as the ratio of the system response to the static deflection $\delta_{st} = \dfrac{f_0}{k}$

thus,

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{\left(1 - \left(\dfrac{w}{w_n}\right)^2\right)^2 + 4\varepsilon^2\left(\dfrac{w}{w_n}\right)^2}}$$

Discussion on the amplification factor:

- For an **undamped system** ($\varepsilon=0$, the amplification factor tends to infinity when $\omega=\omega0$; this is **resonance**.
- An increase in the damping factor leads to a **decrease in the amplification factor**.
- The amplification factor equals 1 (no amplification) when the excitation frequency is zero: $\omega/\omega_0=0$.

- For **very high excitation frequencies**, the system does not respond to the excitation (X=0).

- **Maximum response:** The maximum response of the system to the excitation force is given by:

$$M_{max} = \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\varepsilon\sqrt{1-\varepsilon^2}}$$

At resonance ($w = w_0$), the maximum response of the system to the excitation force becomes:

$$M_r = \frac{1}{2\varepsilon}$$

Facteur de qualité

We saw in the previous paragraph that, when ε is small, the maximum amplification factor is given by:

$$M_{max} = \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\varepsilon\sqrt{1-\varepsilon^2}} = \frac{1}{2\varepsilon} = \left(\frac{X}{\delta_{st}}\right)_{w=w_n}$$

The amplification factor when $w = w_0$ is called the quality factor.

$$Q = \frac{1}{2\varepsilon}.$$

Bandwidth

In the case of a sinusoidal excitation with variable angular frequency ω, and when $\delta < \frac{\omega_0}{\sqrt{2}}$ the bandwidth in angular frequency of the oscillator is defined as the interval: $\Delta\omega = \omega2 - \omega1$ (with $\omega2 > \omega1$) where the angular frequencies $\omega1$ and $\omega2$ correspond to amplitudes $A(\omega1)$ and $A(\omega2)$ such $A(\omega1) = A(\omega2) = \frac{A(\omega r)}{\sqrt{2}}$
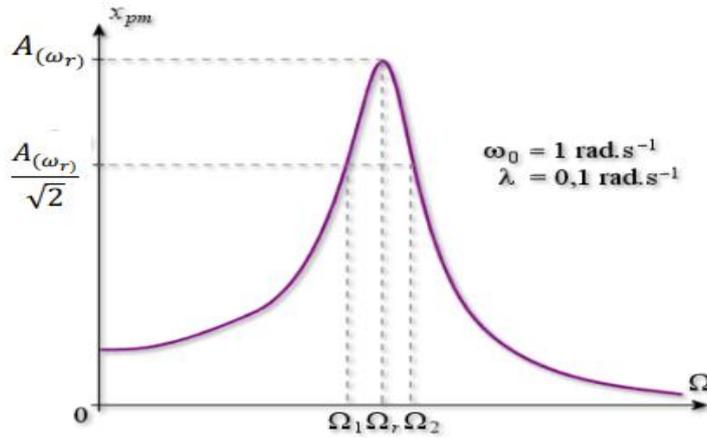
Thus, the bandwidth is given by: $B = \Delta\omega = \omega2 - \omega1 = 2\delta$

Exercise

In the system shown opposite, the ball is point-like and the rod has a total length 3l and negligible mass. The applied force is $F(t) = F0\cos\Omega$.

1. Find the kinetic energy T, the potential energy U, and the dissipation function D.

2. Determine the Lagrangian and then the equation of motion.

3. Find the steady-state solution of the equation of motion (specify its amplitude A and its phase).

4. Deduce the resonance angular frequency Ω_R:

5. Give the cutoff angular frequencies $\Omega1$; $\Omega2$, and the bandwidth B for weak damping ($\lambda <<$ ω_0).

6. Compute Ω_R, B, and the quality factor if m=1kg, k =15N/m, l=0,5m, α=0,5N.s/m, g=10m.s-2.