Linear programming (LP)

- Year 2025/2026
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I Initialization of the simplex algorithm

- The simplex algorithm is a method based on the principle of moving from one feasible solution to another solution that improves the value of the objective function until the optimum is reached.
- However, the initial solution is not always obvious; sometimes an entire initialization phase must be performed.

I Initialization of the simplex algorithm

- For the initialization of the simplex algorithm, i.e., finding a starting solution, we will examine two methods.
  - The two-phase method.
  - The big-M method.

II The two-phase method

- Consider the following linear programming problem:

  Max Z = $c_1x_1+c_2x_2+\ldots+c_nx_n$.

$$\begin{cases} a_{i1}x_1+a_{in}x_2+\ldots+a_{i2}x_n \le b_i, i=1..k \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n = b_i, i=k+1..r \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n \ge b_i, i=r+1..m \\ x_1, x_2, \ldots, x_n \ge 0 \end{cases}$$

II The two-phase method

- The standard form of the problem is as follows:

  Max Z = $c_1x_1+c_2x_2+\ldots+c_nx_n$.

$$\begin{cases} a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n+x_{n+i}=b_i, i=1..k \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n=b_i, i=k+1..r \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n-x_{n+i}=b_i, i=r+1..m \\ x_1, x_2, \ldots, x_n \geq 0 \end{cases}$$

II The two-phase method

- **Problem:** Basic solution not feasible for constraints of type (≥)
- **Solution:** We use a preliminary phase (Phase I)

II.1 Artificial problem

- The constraints therefore become:

$$\begin{cases} a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n+x_{n+i}=b_i, i=1..k \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n+W_s=b_i, i=k+1..r, s=1..r-k \\ a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n-x_{n+i}+W_s=b_i, i=r+1..m, s=r-k+1..m-k \\ x_1, x_2, \ldots, x_n \geq 0 \end{cases}$$

II.1 Artificial problem

- Adding an objective function W that minimizes the sum of the artificial variables:

$$Min\,W = \sum_{s} W_s, s = 1..m-k$$

II.1 Artificial problem

- This problem will be solved using the simplex algorithm.
- The artificial variables w_1, w_2, ..., w_{m-k} are the basic variables of the initial solution, since their values will be non-negative when the variables x_j of the original problem are set to 0.

II.1 Artificial problem

- At the end of phase I
  - (i) If the optimal value min W of the objective function is positive (i.e., min w > 0), i.e if we cannot exclude the artificial variables from the basis, then the original problem does not have solution.
  - (ii) If we manage to exclude artificial variables from the basis with an optimal value of zero (min w = 0), then the initial problem has an optimal solution that will be determined in phase II.

II.2 Phase II

- In the case where min W equals 0, phase II consists of continuing to solve the original problem.
- Which consists of searching for the optimal solution with the simplex algorithm using the initial solution returned by phase 1.
- Considering the simplex table from the last iteration of phase I.

- **Example:** Using the two-phase simplex method, solve the following problem

  Max $Z = x_1 - x_2$

$$\begin{cases} 6x_1 - x_2 \leq 10 \\ x_1 + 5x_2 \geq 4 \\ x_1 + 5x_2 + x_3 = 5, \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

- **Solution:**The artificial problem:
- Max $Z=x_1-x_2$

  Max $W=-w_1$

$$\begin{cases} 6x_1-x_2+x_4=10 \\ x_1+5x_2-x_5+w_1=4 \\ x_1+5x_2+x_3=5, \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Phase 1

Initial table

BV	x_1	x_2	x_3	x_4	x_5	w_1	b_i
x_4	6	-1	0	1	0	0	10
w_1	1	5	0	0	-1	1	4
x_3	1	5	1	0	0	0	5
W	1	5	0	0	-1	0	4
Z	1	-1	0	0	0	0	0

Phase 1

First iteration

BV	x_1	x_2	x_3	x_4	x_5	w_1	b_i
x_4	31/5	0	0	1	-1/5	1/5	54/5
x_2	1/5	1	0	0	-1/5	1/5	4/5
x_3	0	0	1	0	1	-1	1
W	0	0	0	0	0	-1	0
Z	6/5	0	0	0	-1/5	1/5	4/5

Phase 2

Initial table

BV	x_1	x_2	x_3	x_4	x_5	b_i
x_4	31/5	0	0	1	-1/5	54/5
x_2	1/5	1	0	0	-1/5	4/5
x_3	0	0	1	0	1	1
Z	6/5	0	0	0	-1/5	4/5

Phase 2

First iteration

BV	x_1	x_2	x_3	x_4	x_5	b_i
x_4	1	0	0	5/31	-1/31	54/31
x_2	0	1	0	-1/31	-6/31	14/31
x_3	0	0	1	0	1	1
Z	0	0	0	-6/31	-5/31	-40/31

- All coefficients in the objective function line are negative or zero, so the solution is optimal with:

  $x_1^*=54/31$, $x_2^*=14/31$, $x_3^*=1$, $x_4^*=x_5^*=0$, $Z^*=-Z=40/31$

- **Exercise:** solve the following problem using the two-phase simplex method:

  Min $Z=3x_1+2x_2+x_3$

$$\begin{cases} x_1+(4/5)x_3 \geq 4 \\ 2x_1+3x_2+4x_3 \geq 12 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

III. The Big-M method

- The Big-M method consists of combining the two phases into one.
- Starting from problem P, construct a problem (P'), called the M-problem, as follows:

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + x_{n+i} = b_i, i = 1..k \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + W_s = b_i, i = k+1..r, s = 1..r-k \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - x_{n+i} + W_s = b_i, i = r+1..m, s = r-k+1..m-k \\ x_1, x_2, \ldots, x_n \geq 0 \end{cases}$$

III.1 The M-problem

- The objective function becomes:

$$MAX\ Z = \sum C_j X_j - M \sum_s W_s, s = 1..m-k$$

With M (a very large positive number) and W_s representing the artificial variables.

III.2 The M-problem solving

- We solve the new problem using the simplex algorithm.
  - If we manage to exclude all artificial variables from the basis in the optimal solution, we return the solution.
  - If one of the artificial variables remains in the basis and there is no longer a positive index in the objective function, then the constraints are contradictory and the problem does have feasible solutions.

- Example:

Max $Z=3x_1+4x_2+5x_3$

$$\begin{cases} x_1+2x_2+3x_3=10 \\ 2x_1+2x_2+x_3=6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Thank you !