CHAPTER 1

Matrix Analysis

What is a Matrix?

- Intersection of Rows and Columns
- Data Structure Perspective
- Matrix Algebra

$$A\colon \{1,..,\mathrm{n}\} \times \{1,..,\mathrm{m}\} \rightarrow \mathbb{K}$$
$$(i,j) \qquad \rightarrow \; a_{ij}$$

- $1 \leq i \leq n$
  $1 \leq j \leq m$

$a_{ij} \in \mathbb{K}$ real or complex coefficients

<u>Exemple</u>

$$A = \begin{array}{ccc} 1 & 0 & 5 \\ 3 & 2 & 1 \end{array}$$

, function A:

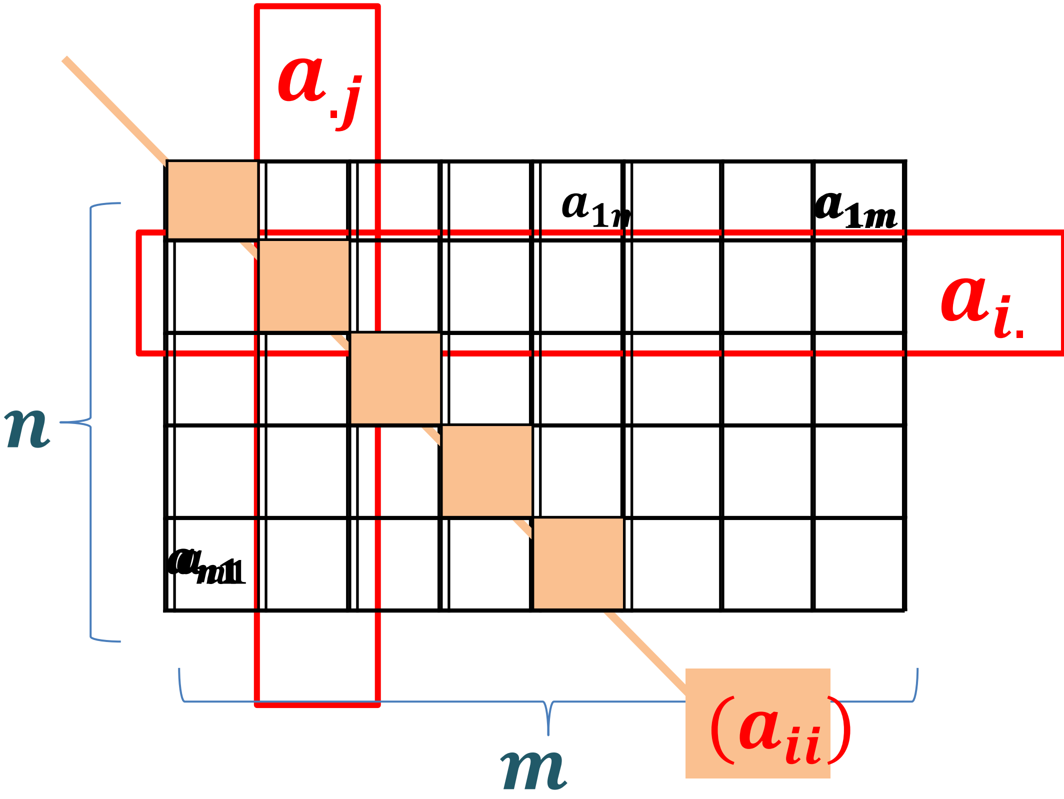
$$(1{,}1) \rightarrow 1 \quad (2{,}1) \rightarrow 3$$
$$(1{,}2) \rightarrow 0 \quad (2{,}2) \rightarrow 2$$
$$(1{,}3) \rightarrow 5 \quad (2{,}3) \rightarrow 1$$

Notations

- $M_n(K)$ $\quad$ $M_{n,m}(K)$

  $1 \leq i \leq n$ $\quad$ $1 \leq i \leq n$

  $1 \leq j \leq n$ $\quad$ $1 \leq j \leq m$



- $(a_{i.})$: 1xm vector
- $(a_{.j})$: nx1 vector
- (a_{ii}) : min(n,m)x1 vector

Questions

- Does this function define a matrix representation?

$(1,1) \rightarrow 1 \qquad (2,1) \rightarrow 2i + 4$

- A: $(1,2) \rightarrow i + 3$

$(1,3) \rightarrow 5 \qquad (2,3) \rightarrow 1$

$(1,1) \rightarrow 0 \qquad (2,1) \rightarrow -2$

- A: $(1,2) \rightarrow 1 \qquad (2,2) \rightarrow 0$

$(1,3) \rightarrow -1 \qquad (2,3) \rightarrow 1$

Notations

- $(A^t)_{ji}$ **transposed** matrix of A

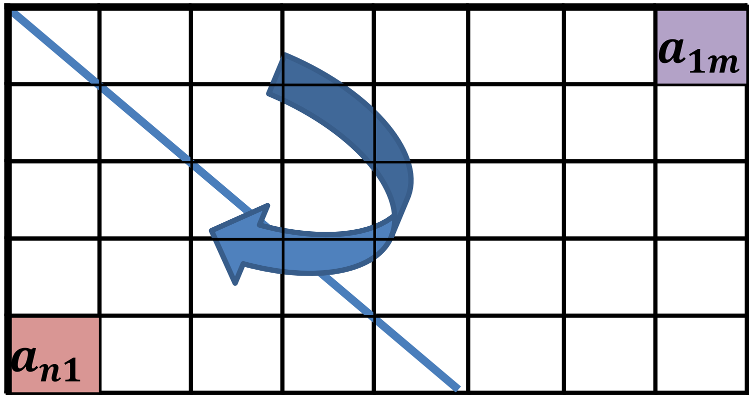
  $(A^t)^t = A$

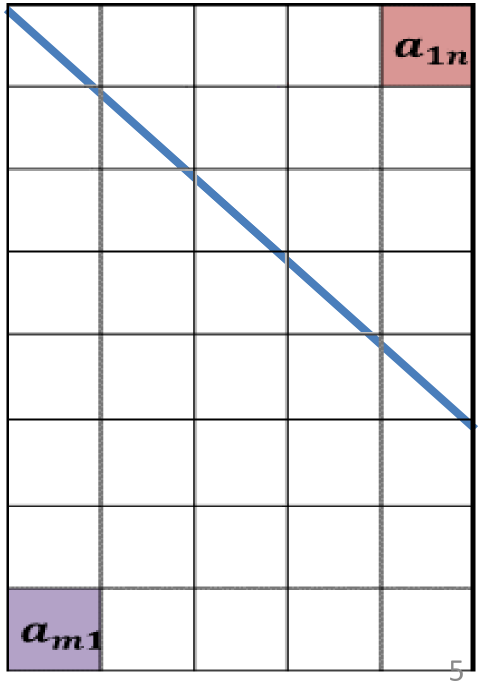
- $(A^*)_{ji}$ Hermitian **adjoint** matrix of $(A)_{ij}$

  $$(A^*)_{ji} = (\overline{A^t})_{ji}$$

  $(A^*)^* = A$

- $A_{1,m}(K)$
- $A_{n,1}(K)$





Notations

Example 1 :

$$\begin{pmatrix} 1 & 2 \\ 0 & 5 \\ 4 & 7 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 5 & 7 \end{pmatrix}$$

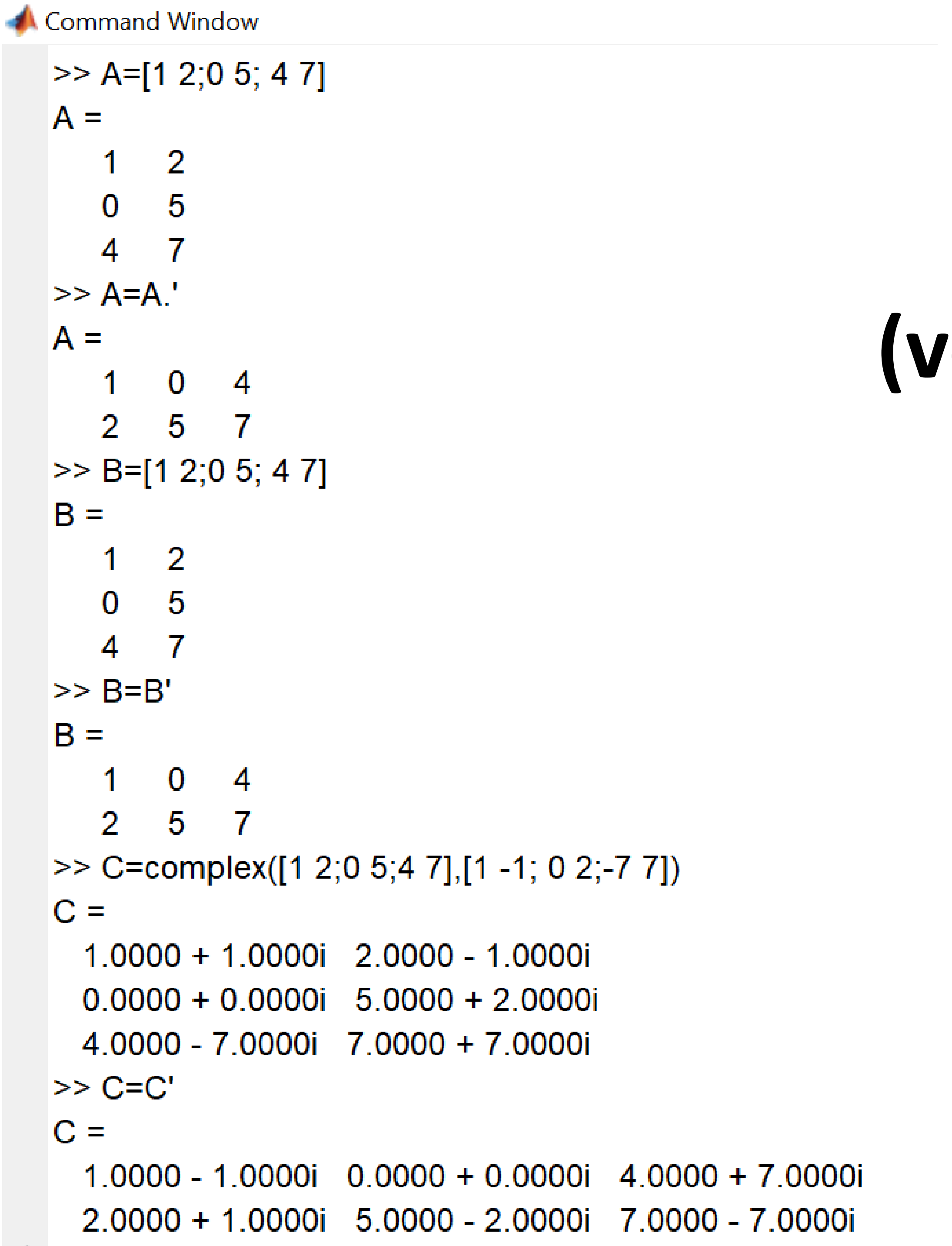
Example 2 :

$$\begin{pmatrix} 1 & 2 \\ 0 & 5 \\ 4 & 7 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 5 & 7 \end{pmatrix}$$

Example 3 :

$$\begin{pmatrix} 1+i & 2-3i \\ 0 & 5+2i \\ 4-7i & 7 \end{pmatrix}^* = \begin{pmatrix} 1-i & 0 & 4+7i \\ 2+3i & 5-2i & 7 \end{pmatrix}$$

via MATLAB (vectorization)



```
>> A=[1 2;0 5; 4 7]
A =
     1     2
     0     5
     4     7
>> A=A.'
A =
     1     0     4
     2     5     7
>> B=[1 2;0 5; 4 7]
B =
     1     2
     0     5
     4     7
>> B=B'
B =
     1     0     4
     2     5     7
>> C=complex([1 2;0 5;4 7],[1 -1; 0 2;-7 7])
C =
   1.0000 + 1.0000i   2.0000 - 1.0000i
   0.0000 + 0.0000i   5.0000 + 2.0000i
   4.0000 - 7.0000i   7.0000 + 7.0000i
>> C=C'
C =
   1.0000 - 1.0000i   0.0000 + 0.0000i   4.0000 + 7.0000i
   2.0000 + 1.0000i   5.0000 - 2.0000i   7.0000 - 7.0000i
```

Operations on Matrices

Let $\mathrm{A}, \mathrm{B}, \mathrm{C} \in M_{n,m}(K)$,

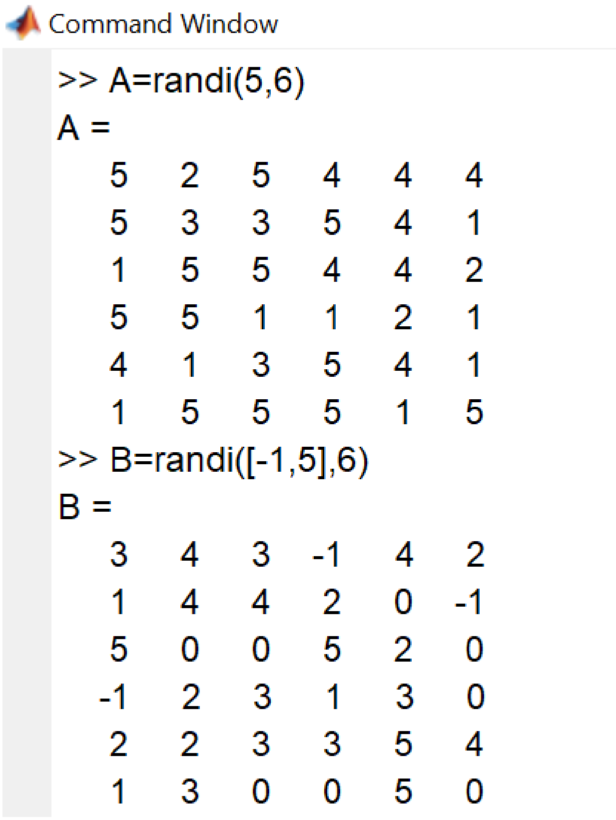
$$\forall (i, j) \in \{1, .., \mathrm{n}\} \times \{1, .., \mathrm{m}\}$$

- Equality $a_{ij} = b_{ij} \Leftrightarrow A = B$
- Addition $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij}$
  - $0_{n,m}$ **neutral element for addition**
  - $A + B = B + A$ **commutativity**
  - $(A + B) + C = A + (B + C)$ **associativity**

  Transpose (conjugate transpose) distributes over matrix addition

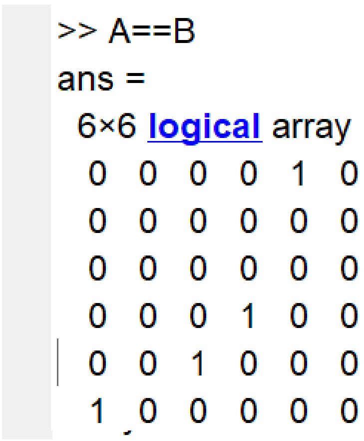
  - $(A + B)^t = (A)^t + (B)^t$
  - $(A + B)^* = (A)^* + (B)^*$

Operations via MATLAB

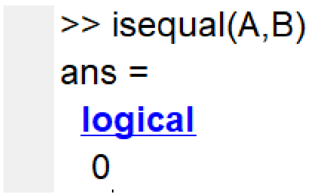


Command Window

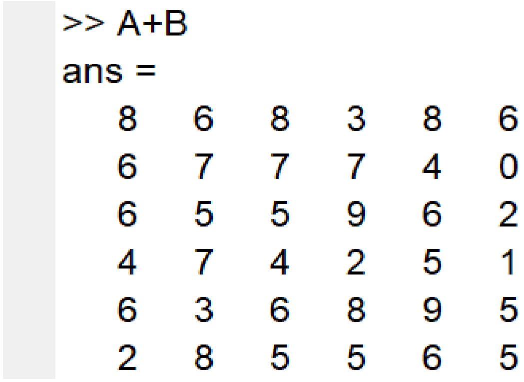
```
>> A=randi(5,6)
A =
     5     2     5     4     4     4
     5     3     3     5     4     1
     1     5     5     4     4     2
     5     5     1     1     2     1
     4     1     3     5     4     1
     1     5     5     5     1     5
>> B=randi([-1,5],6)
B =
     3     4     3    -1     4     2
     1     4     4     2     0    -1
     5     0     0     5     2     0
    -1     2     3     1     3     0
     2     2     3     3     5     4
     1     3     0     0     5     0
```



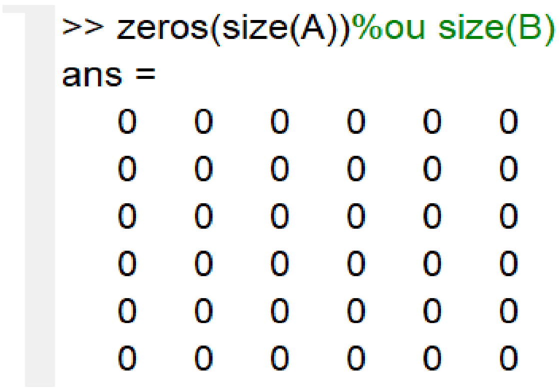
```
>> A==B
ans =
  6×6 logical array
   0   0   0   0   1   0
   0   0   0   0   0   0
   0   0   0   0   0   0
   0   0   0   1   0   0
   0   0   1   0   0   0
   1   0   0   0   0   0
```



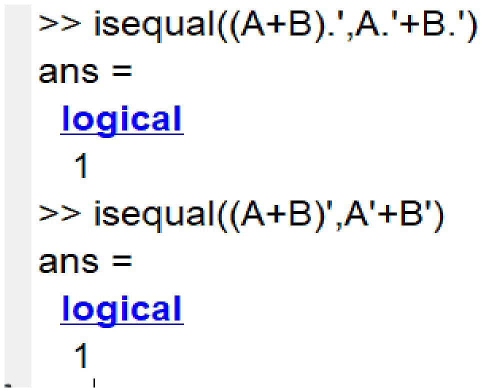
```
>> isequal(A,B)
ans =
  logical
   0
```



```
>> A+B
ans =
     8     6     8     3     8     6
     6     7     7     7     4     0
     6     5     5     9     6     2
     4     7     4     2     5     1
     6     3     6     8     9     5
     2     8     5     5     6     5
```



```
>> zeros(size(A))%ou size(B)
ans =
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
```



```
>> isequal((A+B).',A.'+B.')
ans =
  logical
   1
>> isequal((A+B)',A'+B')
ans =
  logical
   1
```

Operations on Matrices

- Scalar Multiplication

$$C = \lambda A \iff c_{ij} = \lambda \cdot a_{ij}$$

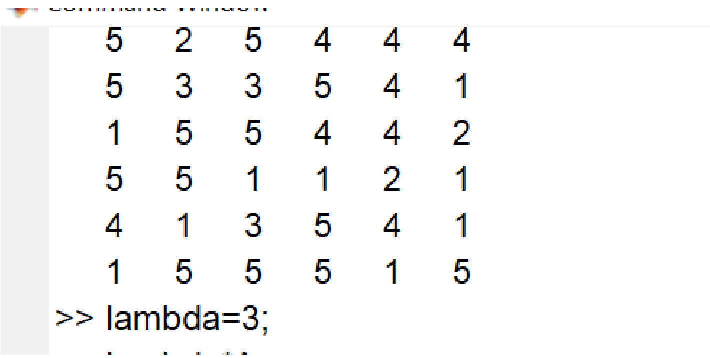
$$(\lambda A)^t = \lambda A^t$$

$$(\lambda A)^* = \bar{\lambda} A^*$$

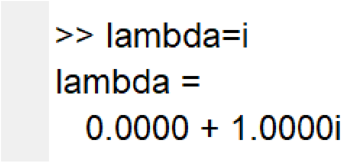
$$\lambda(A + B) = \lambda A + \lambda B$$

$$(\lambda + \mu)A = \lambda A + \mu A$$

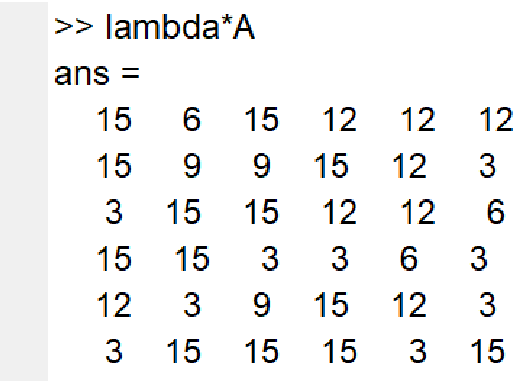
Operations via MATLAB



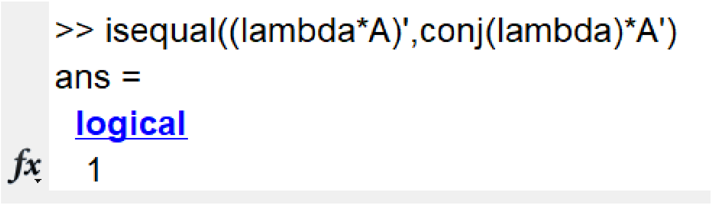
```
     5     2     5     4     4     4
     5     3     3     5     4     1
     1     5     5     4     4     2
     5     5     1     1     2     1
     4     1     3     5     4     1
     1     5     5     5     1     5
>> lambda=3;
```



```
>> lambda=i
lambda =
   0.0000 + 1.0000i
```

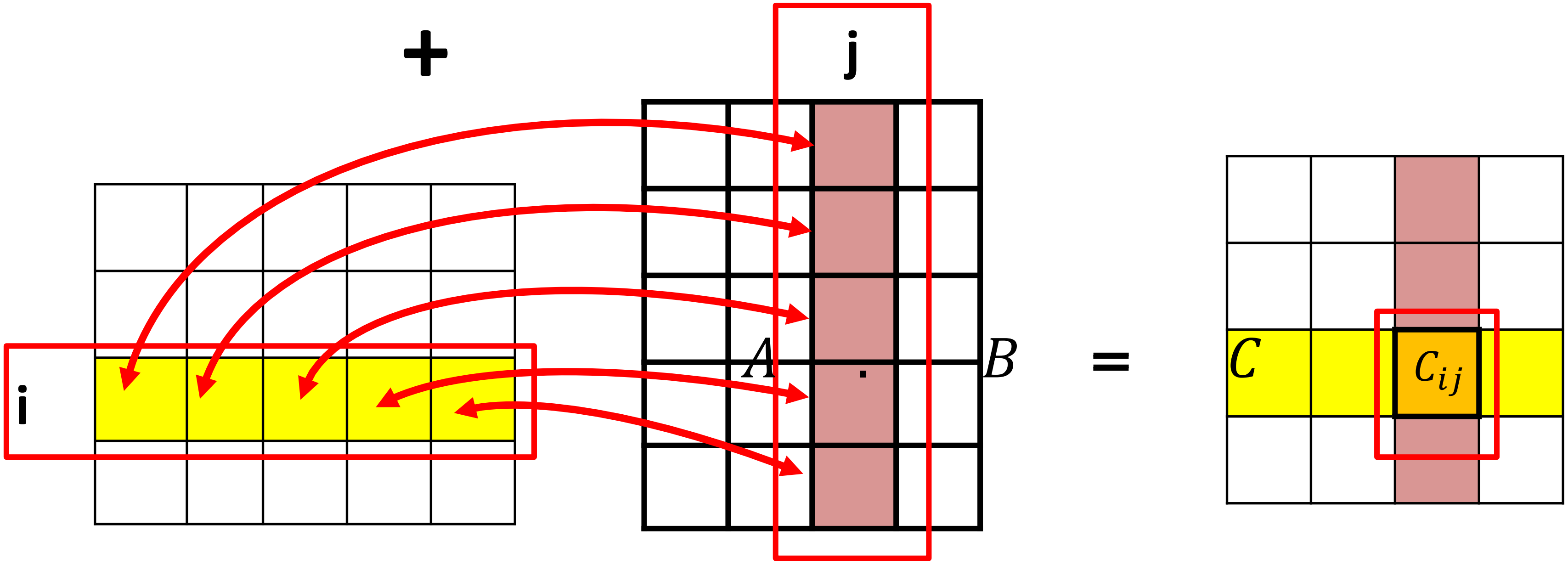


```
>> lambda*A
ans =
    15     6    15    12    12    12
    15     9     9    15    12     3
     3    15    15    12    12     6
    15    15     3     3     6     3
    12     3     9    15    12     3
     3    15    15    15     3    15
```



```
>> isequal((lambda*A)',conj(lambda)*A')
ans =
  logical
   1
```

Matrix Multiplication



$A \in M_{n,p}(K)$ $B \in M_{p,m}(K)$ $C \in M_{n,m}(K)$

- $C = A\,B \iff c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}$

Matrix Multiplication

Example:

- if A is of order 3 and B of size 3 ×2
- then A ×B is of order 3 ×2

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 2 \\ 6 & 11 \end{pmatrix}$$

- MATLAB : >> A*B

Multiplicative Identity Matrix

- $I_n = \delta_{ij} = \begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{matrix}$

- $A\,.\,I = A$

- $A \in M_{n,\boldsymbol{p}}(K),\ I_p \qquad A\,.\,I = A$
- $A \in M_{n,\boldsymbol{p}}(K),\ I_n \qquad I.\,A = A$

Properties

- $A + B = B + A\,,$

  $$(A + B)^t = A^t + B^t$$

  $$(A + B)^* = A^* + B^*$$

  $$A + (B + C) = (A + B) + C$$

- $A\,.\,B \quad \neq \quad B.\,A$

  $$(A \cdot B)^t = A^t \cdot B^t$$

  $$(A \cdot B)^* = A^* \cdot B^*$$

- $A\,.\,(B.\,C) = (A\,.\,B).\,C$
- A.(B+C)=A.B+A.C
- $(A + B)C = AC + BC$

Properties

- $A = 0 \quad$ and/or $\quad B = 0 \qquad \Rightarrow A\,.\,B = 0$
- $A\,.\,B = 0 \;\not\Rightarrow A = 0 \; or \; B = 0$

- $A^p = A \cdot A \cdot \; \dots \cdot A \quad p$ fois
- Exemple

$$\mathrm{A} = \begin{matrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{matrix} \qquad A^p = \begin{matrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{matrix}$$

$$A^5 = \begin{matrix} 1 & 0 & 31 \\ 0 & -1 & 0 \\ 0 & 0 & 32 \end{matrix} \qquad A^0 = I$$

The *inverse of a matrix*

Let $A \in M_n(K)$,

- Existence

  $\exists ?\ A^{-1}$ tq $\quad A \cdot A^{-1} = I_n$

- The inverse of A is denoted by A^{-1}
- If $\exists\ A^{-1}$ then A is said to be **non-singular** or **invertible**.

- <u>Propreties</u>

  $(A^{-1})^{-1} = A$

  $(A^{-1})^{T} = (A^{T})^{-1}$

  $(AB)^{-1} = B^{-1}A^{-1}$

  $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

The *inverse of a matrix*

Examples :

- $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **invertible**?

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} a = 1 - 2c & c = 0 \\ b = -2d & d = 1/3 \end{matrix}$$

$$A^{-1} = \begin{matrix} 1 & -2/3 \\ 0 & 1/3 \end{matrix}$$

A **non-singular**

- $A = \begin{matrix} 3 & 0 \\ 5 & 0 \end{matrix} \qquad \nexists\, A^{-1}$

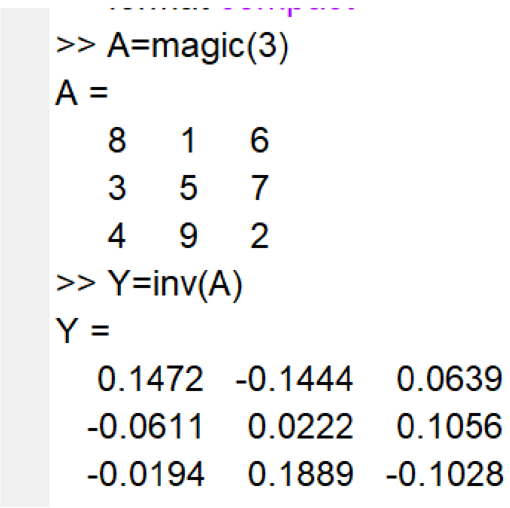
A is singular,

non-invertible

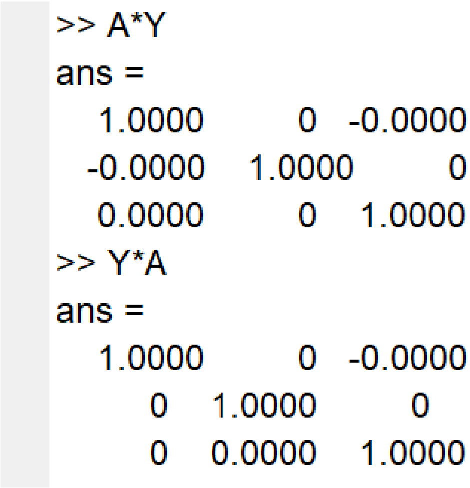
- $I^{-1} = I$
- **The zero matrix is never invertible**.

The *inverse with MATLAB*

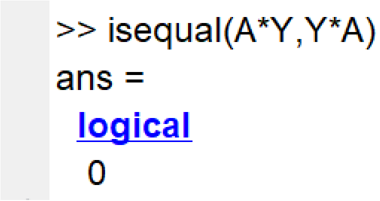
- $A \in M_n(K)$,
- R=inv(A)
- A^(-1)



```
>> A=magic(3)
A =
     8     1     6
     3     5     7
     4     9     2
>> Y=inv(A)
Y =
    0.1472   -0.1444    0.0639
   -0.0611    0.0222    0.1056
   -0.0194    0.1889   -0.1028
```

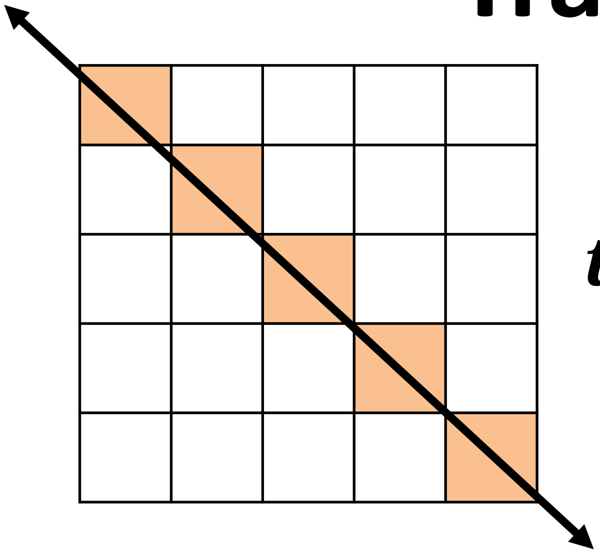


```
>> A*Y
ans =
    1.0000         0   -0.0000
   -0.0000    1.0000         0
    0.0000         0    1.0000
>> Y*A
ans =
    1.0000         0   -0.0000
         0    1.0000         0
         0    0.0000    1.0000
```



```
>> isequal(A*Y,Y*A)
ans =
  logical
   0
```

Trace of a Matrix



$$\boldsymbol{tr(A) = \sum a_{ii}}$$

- $tr(\alpha A + B) = \alpha \cdot tr(A) + tr(B)$
- Let $A \in \mathcal{M}_{n,m}(\mathbb{R}), B \in \mathcal{M}_{m,n}(\mathbb{R})$, then
$$tr(A \cdot B) = tr(B \cdot A)$$

Trace of a matrix (MATLAB)

```
>> sum(diag(A))
```

```
>> isequal (sum(diag(A*B)),sum(diag(B*A)))
```

```
>>isequal(sum(diag(A)), sum(diag(A’)))
```

Determinant of a Matrix

$$\det : \begin{cases} M_n(K) & \rightarrow & K \\ (A_{\boldsymbol{i,j}}) & \rightarrow & y \end{cases}$$

Example :

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ → $\det A = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$

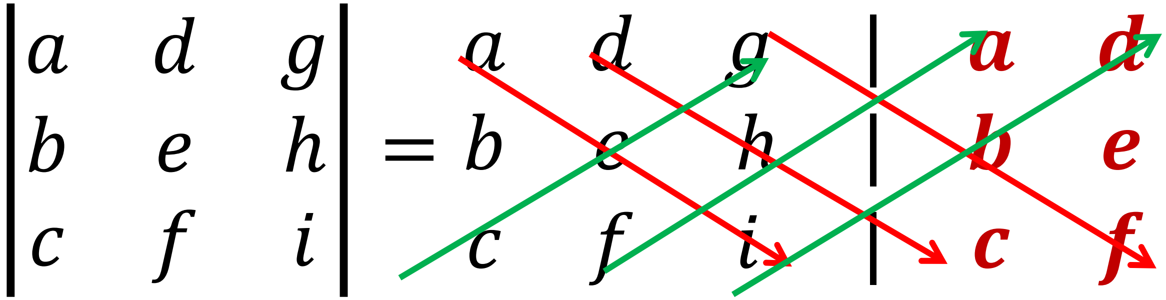
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ →

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - b.c$$

Computation of the determinant

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = ??$$

Sarrus' Rule



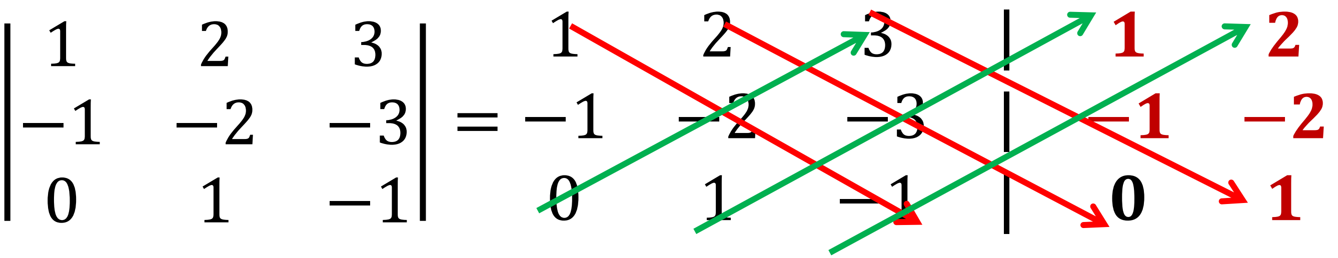
$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{array}{ccc|cc} a & d & g & a & d \\ b & e & h & b & e \\ c & f & i & c & f \end{array}$$

$$= aei + dhc + gbf - ceg - fha - ibd$$

Computation of the determinant

$$\begin{vmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 0 & 1 & -1\end{vmatrix} = ??$$

Sarrus' Rule



$$\begin{vmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 0 & 1 & -1\end{vmatrix} = \begin{array}{ccc|cc}1 & 2 & 3 & \mathbf{1} & \mathbf{2}\\ -1 & -2 & -3 & \mathbf{-1} & \mathbf{-2}\\ 0 & 1 & -1 & \mathbf{0} & \mathbf{1}\end{array}$$

$$\begin{aligned}&= \mathbf{1(-2)(-1) + 2(-3)0 + 3(-1)1}\\ &\quad - \mathbf{[0(-2)3 + 1(-3)1 + (-1)(-1)2] = 0}\end{aligned}$$

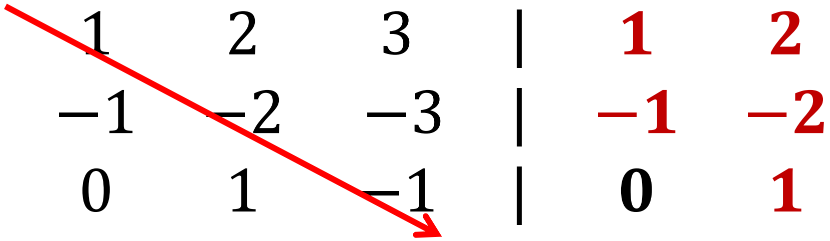
Sarrus' Rule via MATLAB

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

```
if isequal(size(M),[3,3])
```

$$\begin{array}{ccc|cc} 1 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ -1 & -2 & -3 & \mathbf{-1} & \mathbf{-2} \\ 0 & 1 & -1 & \mathbf{0} & \mathbf{1} \end{array}$$

```
M=[M,M(:,1:2)]
```



$$\begin{array}{ccc|cc} 1 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ -1 & -2 & -3 & \mathbf{-1} & \mathbf{-2} \\ 0 & 1 & -1 & \mathbf{0} & \mathbf{1} \end{array}$$

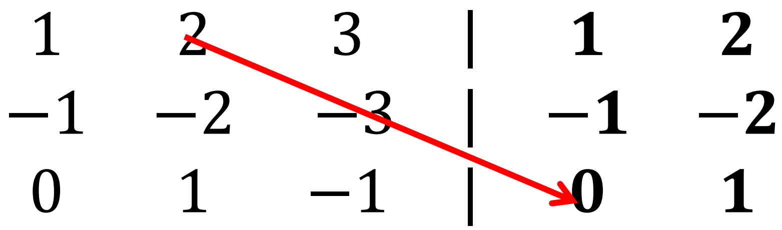
```
D(1)=prod(diag(M))
```

Or

```
D(1)=prod(diag(M(:,1:end)))
```

2

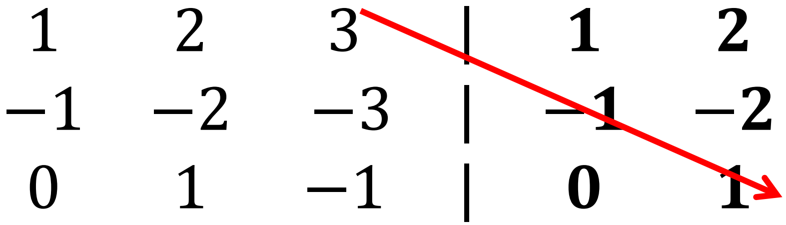
Sarrus' Rule via MATLAB



$$\begin{array}{ccc|cc} 1 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ -1 & -2 & -3 & \mathbf{-1} & \mathbf{-2} \\ 0 & 1 & -1 & \mathbf{0} & \mathbf{1} \end{array}$$

D(2)=prod(diag(M(:,2:end)))

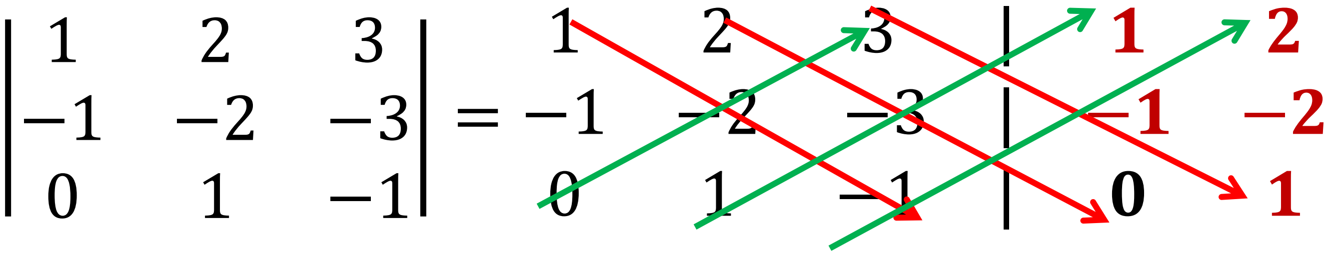
2	**0**



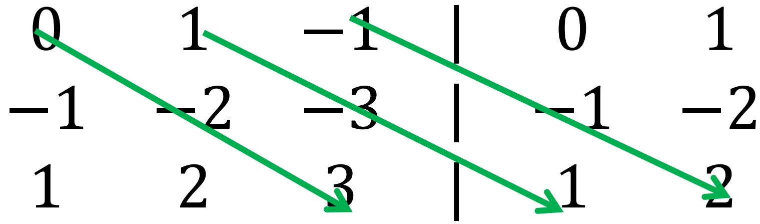
$$\begin{array}{ccc|cc} 1 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ -1 & -2 & -3 & \mathbf{-1} & \mathbf{-2} \\ 0 & 1 & -1 & \mathbf{0} & \mathbf{1} \end{array}$$

D(3)=prod(diag(M(:,3:end)))

2	**0**	**-3**



$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & 1 & -1 \end{vmatrix} = \begin{array}{ccc|cc} 1 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ -1 & -2 & -3 & \mathbf{-1} & \mathbf{-2} \\ 0 & 1 & -1 & \mathbf{0} & \mathbf{1} \end{array}$$



$$\begin{array}{ccc|cc} 0 & 1 & -1 & 0 & 1 \\ -1 & -2 & -3 & -1 & -2 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

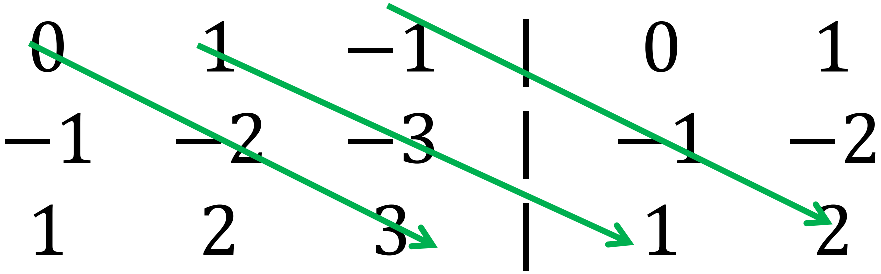
flipud(M(:,1:end))

Sarrus' Rule via MATLAB

`DD(1)=prod(diag(flipud(M(:,1:end))))`

`DD(2)=prod(diag(flipud(M(:,2:end))))`

`DD(3)=prod(diag(flipud(M(:,3:end))))`



$$\begin{array}{ccc|cc} 0 & 1 & -1 & 0 & 1 \\ -1 & -2 & -3 & -1 & -2 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

2	**0**	**-3**

0	**-3**	**2**

DET = sum(D) - sum(DD)

Sarrus' Rule via MATLAB

```
M=[M,M(:,1:2)]

D(1)=prod(diag(M(:,1:end)))

D(2)=prod(diag(M(:,2:end)))

D(3)=prod(diag(M(:,3:end)))
                            DD(1)=prod(diag(flipud(M(:,1:end))))

                            DD(2)=prod(diag(flipud(M(:,2:end))))

                            DD(3)=prod(diag(flipud(M(:,3:end))))

                            DET=sum(D)-sum(DD)
```

Sarrus' Rule via MATLAB

```
function[DET]=sarrus(M)
if isequal(size(M),[3,3])
   M=concat(M)
   [D,DD]=les_prod(M)
   DET=0
   for (i=1:length(D))
      DET=DET+ D(i)-DD(i)
   end
else fprintf('on ne peut appliquer Sarrus’)
end,end
```

```
function[M]=concat(A)
M=A
for j=1:2
for i=1:size(A,1)
   M(i,size(A,2)+j)=M(i,j)
end,end
```

```
>>A=magic(3)
>>sarrus(A)
```

```
function[d,dd]=les_prod(M)
for k=1:3
   d(k)=prod_diag(M(:,k:end))
   dd(k)=prod_antidiag(M(:,k:end))
end,end
```

```
function[p]=prod_diag(A)
   p=1
   for i=1:min(size(A))%ou 3
      p=p*A(i,i)
   end,
end
```

```
function[p]=prod_antidiag(A)
   p=1
   for i=min(size(A)):-1:1
      p=p*A(i,min(size(A))-i+1)
   end,         %ou 3
end
```

Computation of the determinant

Laplace expansion
(cofactor expansion)

$A \in M_{\boldsymbol{n}}(K)$

minor matrix $(\boldsymbol{A_{ij}})$

cofactor of element $\boldsymbol{a_{ij}}$ **,** $\boldsymbol{Cof_{ij}}$

Laplace expansion

Let $A = \begin{matrix} 4 & 0 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{matrix} \in M_{\mathbf{4}}(\boldsymbol{R})$

Minor matrices $(\boldsymbol{A_{ij}})$

$$(A_{12}) = \begin{matrix} 4 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{matrix} \qquad (A_{22}) = \begin{matrix} 4 & 3 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{matrix} \qquad (A_{13}) = \begin{matrix} 4 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 3 \end{matrix}$$

$$(A_{32}) = \begin{matrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{matrix} \qquad (A_{42}) = \begin{matrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & -1 \end{matrix} \qquad (A_{34}), (A_{31}), (A_{43}), \ldots$$

Laplace expansion

$\boldsymbol{Cof}_{\boldsymbol{ij}}$, Cofactor of element $\boldsymbol{a_{ij}}$

$$Cof_{ij} = (-1)^{i+j} \cdot \left|(A_{ij})\right|$$

$$Cof_{12} = (-1)^{1+2} \ |(A_{12})|$$

$$Cof_{22} = (-1)^{2+2} \ |(A_{22})\ |$$

$$Cof_{32} = (-1)^{3+2} \ |(A_{32})\ |$$

$$Cof_{42} = (-1)^{4+2} \ |(A_{42})\ |$$

Laplace expansion

The expansion

$$A_{i{\color{red}\boldsymbol{j}}} = \begin{array}{cccc} a & \cancel{d} & \cdots & x \\ b & \cancel{e} & \cdots & y \\ \vdots & \cancel{\vdots} & \ddots & \vdots \\ c & \cancel{f} & \cdots & z \\ & \uparrow & & \\ & {\color{red}\boldsymbol{j}} & & \end{array}$$

$$A_{{\color{red}\boldsymbol{i}}j} = \begin{array}{ccccc} a & d & \cdots & x & \\ \cancel{b} & \cancel{e} & \cancel{\cdots} & \cancel{y} & \leftarrow {\color{red}\boldsymbol{i}} \\ \vdots & \vdots & \ddots & \vdots & \\ c & f & \cdots & z & \end{array}$$

Laplace expansion

Steps :

- Fix expansion index i or j
- **Determine** coresponding minor matrices ($\boldsymbol{A_{ij}}$)
- calculate corresponding cofactors of $\boldsymbol{a_{ij}}$, Cof_{ij}

$$\det A = \sum_{\substack{i=1 \\ or \\ j=1}}^{n} a_{ij} \, . \, Cof_{ij}$$

Laplace expansion

Example :

$$A = \begin{array}{c|c|cc} \hline 4 & 0 & 3 & 1 \\ \hline 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{array}$$

j=2 or i=1

$$(A_{12}) = \begin{matrix} 4 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{matrix}$$

$$(A_{22}) = \begin{matrix} 4 & 3 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{matrix}$$

$$(A_{32}) = \begin{matrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{matrix}$$

$$(A_{42}) = \begin{matrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$$

$$(A_{11}) = \begin{matrix} 2 & 1 & 0 \\ 3 & 1 & -1 \\ 0 & 2 & 3 \end{matrix}$$

$$(A_{12}) = \begin{matrix} 4 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{matrix}$$

$$(A_{13}) = \begin{matrix} 4 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 3 \end{matrix}$$

$$(A_{14}) = \begin{matrix} 4 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{matrix}$$

Example :

$$Cof_{ij} = (-1)^{i+j}\ .\left|(A_{ij})\right|$$

$$A = \begin{array}{c|c|cc} 4 & 0 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{array}$$

$$|(A_{12})| = \begin{vmatrix} 4 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 4.1.3 + 1.(-1).1 + 0.0.2 - 1.1.0$$

$$-2.(-1).4 - 3.0.1 = 19$$

$$\boxed{Cof_{12} = (-1)^{1+2}\ .|(A_{12})| = -19}$$

$$|(A_{22})| = \begin{vmatrix} 4 & 3 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 4.1.3 + 3.(-1).1 + 1.0.2 - 1.1.1 -$$

$$2.(-1).4\ - 3.0.3 = 16$$

$$\boxed{Cof_{22} = (-1)^{2+2}\ .|(A_{22})| = 16}$$

Example :

$$A = \begin{array}{c|c|cc} 4 & 0 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{array}$$

$$|(A_{32})| = \begin{vmatrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 4.3.1 + 3.0.1 + 1.4.2 \ - 1.1.1$$

$$-2.0.4 - 3.4.3 = -17$$

$$\boxed{Cof_{32} = (-1)^{3+2} \ .|(A_{32})| = 17}$$

$$|(A_{42})| = \begin{vmatrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 4.1.(-1) + 3.0.0 + 1.4.1 \ - 0.1.1$$

$$-1.0.4 - (-1).4.3 = 12$$

$$\boxed{Cof_{42} = (-1)^{4+2} \ .|(A_{42})| = 12}$$

Example :

$$\det A = \sum_{i=1}^{n} a_{ij} \, . \, Cof_{ij} \qquad A = \begin{matrix} 4 & 0 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{matrix}$$

$$\det A = \cancel{0 \cdot Cof_{12}} + 2 \cdot Cof_{22} + 3 \cdot Cof_{32} + \cancel{0 \cdot Cof_{42}}$$

$$Cof_{22} = 16 \qquad Cof_{32} = 17$$

$$\boxed{\det A = 83}$$

Laplace expansion

$$A = \begin{array}{cccc} 4 & 0 & 3 & 1 \\ \hline 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{array} \quad \leftarrow \boldsymbol{i}$$

$$\det A = \sum_{j=1}^{n} a_{ij}\, Cof_{ij}$$

$$= 4. Cof_{11} + 3.\ Cof_{13} + 1.\ Cof_{14}$$

$$|A_{11}| = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & -1 \\ 0 & 2 & 3 \end{vmatrix} = \qquad 2 \times (3+2) - 1 \times 9 = 1$$

$$= \quad 2\,.\,(-1^{1+1}).\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1.(-1)^{1+2}.\begin{vmatrix} 3 & -1 \\ 0 & 3 \end{vmatrix}$$

$$|A_{13}| = \begin{vmatrix} 4 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 3 \end{vmatrix} = \qquad 4 \times 9 - 2 \times 1 = 34$$

$$= \quad 4\,.\,(-1^{1+1}).\begin{vmatrix} 3 & -1 \\ 0 & 3 \end{vmatrix} + 2.(-1)^{1+2}.\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix}$$

$$|A_{14}| = \begin{vmatrix} 4 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \qquad -2 \times (-1) + 3 \times (8-1) = 23$$

$$= \quad 2\,.\,(-1^{1+2}).\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 3.(-1)^{2+2}.\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}$$

Laplace expansion

$$A = \begin{matrix} \overline{4} & \overline{0} & \overline{3} & \overline{1} & \leftarrow \boldsymbol{i} \\ 4 & 2 & 1 & 0 & \\ 0 & 3 & 1 & -1 & \\ 1 & 0 & 2 & 3 & \end{matrix}$$

$$Cof_{11} = (-1^{1+1}).\,|A_{11}| = 1$$

$$Cof_{13} = (-1^{1+3}).\,|A_{13}| = 34$$

$$Cof_{14} = (-1^{1+4}).\,|A_{14}| = -23$$

$$\det A = \sum_{j=1}^{n} a_{ij}\, Cof_{ij} \;= 4.\,Cof_{11} + 3.\,Cof_{13} + 1.\,Cof_{14}$$

$$= 83$$

Laplace via MATLAB

```
function[d]=determ(M)
if isequal(size(M),[2,2])
    d=M(1,1)*M(2,2)-M(2,1)*M(1,2)
else
     i=1
    d=0
    %the expansion
    for j=1:size(M,2)
        d=d+M(i,j)*cofacteur(M,i,j)
    end
end
end
```

```
function[cof]=cofacteur(M,i,j)
cof =(-1)^(i+j) * det_mineur(M,i,j)
end
```

```
function[mi]=det_mineur(M,i,j)
M(i,:)=[]
M(:,j)=[]
mi=determ(M)
end
```

Determinant properties

$$\det A \cdot B = \det A \cdot \det B$$

$$\det\ A^{-1} = \frac{1}{\det A}$$

$$\det\ A^{t} = \det A$$

$$det\ A^{*} = \overline{\det A}$$

$$\det \lambda A = \ \lambda^{n}\ \det A$$

$$\det\ I_{n} = 1$$

Determinant properties

$$\text{Let : A} = \begin{array}{cccccccc} C_1 & \cdots & C_k & \cdots & C_l & \cdots & C_n & \\ a_{11} & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & L_1 \\ a_{21} & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & L_2 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & L_n \end{array}$$

$det\, A = 0$ for 3 cases

1. $A = 0_n$

2. If $\exists\ 1 \leq k \leq l \leq n$ where $C_k = C_l$ or $L_k = L_l$

3. Si $\exists\ k$ tel que $C_k = \sum_{\substack{i=1 \\ i \neq k}}^{\mathrm{n}} \lambda_i C_i$ or $L_k = \sum_{\substack{i=1 \\ i \neq k}}^{\mathrm{n}} \lambda_i L_i$

Determinant properties

Examples

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \Rightarrow \det A = 0 \text{ since } C_2 = C_1$$

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 0 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix} \Rightarrow \det A = 0 \text{ since } C_3 = C_1 - 2C_2$$

Matrix Inverse

Let $A \in M_n(K)$,

- Existence

  $\exists?\ A^{-1}$ tq $\quad A \cdot A^{-1} = I_n$

- The inverse of A is denoted by A^{-1}
- If $\exists\ A^{-1}$ then A is said to be **non-singular** or **invertible**.

Matrix Inverse computation

Let : $A, C \in M_n(K)$

$C = Com(A) =$ **comatrice** of A

$$= \left(Cof_{ij}\right)_{\substack{i=1..n \\ j=1..n}}$$

$$cof_{ij} = (-1)^{i+j} \det(A_{ij})$$

$$A^{-1} = \frac{1}{\det A} C^t$$

$$\det A \neq 0$$

Inverse computation

$$\boldsymbol{A} = \begin{matrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{matrix}$$

,

$$\det A = 1.(-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 1.(-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0.(-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 2$$

$\Rightarrow \quad A$ Non singular

$$Cof_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$Cof_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$Cof_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$Cof_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$Cof_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$Cof_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$Cof_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$Cof_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$Cof_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}^t$$

Inverse computation

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \; ,$$

$\det A = 0 \;\; \Rightarrow A$ non invertible

$\det B = 0 \;\; \Rightarrow B$ non invertible

B= $\begin{bmatrix} 4 & 3 & -2 \\ 0 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix}$

Inverse via MATLAB

```
function[d]=determ(M)
if isequal(size(M),[2,2])
   d=M(1,1)*M(2,2)-M(2,1)*M(1,2)
else
   i=1  % développer sur la 1ere ligne
   d=0 %initialiser le déterminant
   %le  développement
   for j=1:size(M,2)
      d=d+M(i,j)*cofacteur(M,i,j) end
end
end
 function[cof]=cofacteur(M,i,j)
cof=(-1)^(i+j) * det_mineur(M,i,j)
end
function[mi]=det_mineur(M,i,j)
M(i,:)=[]
M(:,j)=[]
mi=determ(M)
end
```

```
function[inve]=inverse(M)
for i=1:size(M,1)
   for j=1:size(M,2)
      c(i,j)=cofacteur(M,i,j)
      inve(j,i)=c(i,j)
   end
end
inve=(1/detrm(A))*inve
end
```

- How to ameliorate?

Special matrices

Diagonal matrix

$A \in M_n(K)$

$\forall (i,j)$, such that $i \neq j, a_{ij} = 0$

2	0	0	0
0	4	0	0
0	0	0	0
0	0	0	7

- Addition $A + B$ is diagonal
- Product $A \times B$ is diagonal
- $\det A = \prod_{i=1}^{n} a_{ii}$,
- $\forall i, a_{ii} \neq 0 \Leftrightarrow A$ invertible
- $A^{-1}, i \neq j,\ a_{ij}^{-1} = 0,$

  $i = j,\ a_{ij}^{-1} = \dfrac{1}{a_{ij}}$

$\frac{1}{a_{11}}$	0	0	0
0	$\frac{1}{a_{22}}$	0	0
0	0	$\ddots$	0
0	0	0	$\frac{1}{a_{nn}}$

Special matrices

Diagonal Matrix

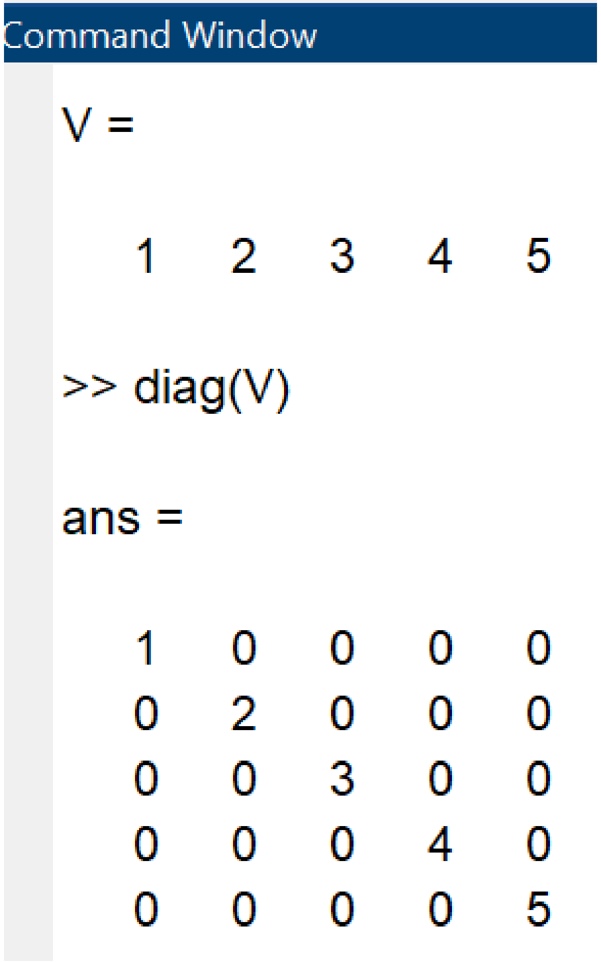
- Example

2	**0**	**0**	**0**
0	**4**	**0**	**0**
0	**0**	**2**	**0**
0	**0**	**0**	**7**

1/2	**0**	**0**	**0**
0	**1/4**	**0**	**0**
0	**0**	**1/2**	**0**
0	**0**	**0**	**1/7**

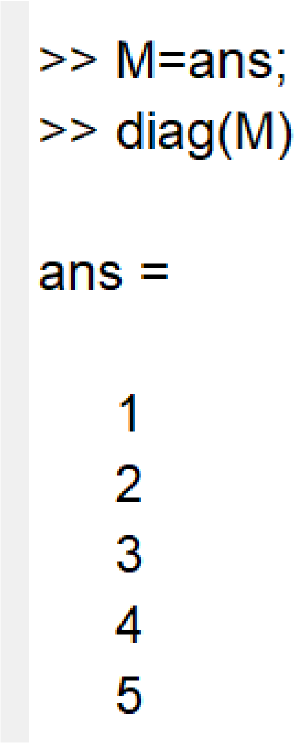
Diagonal Matrix

Creation

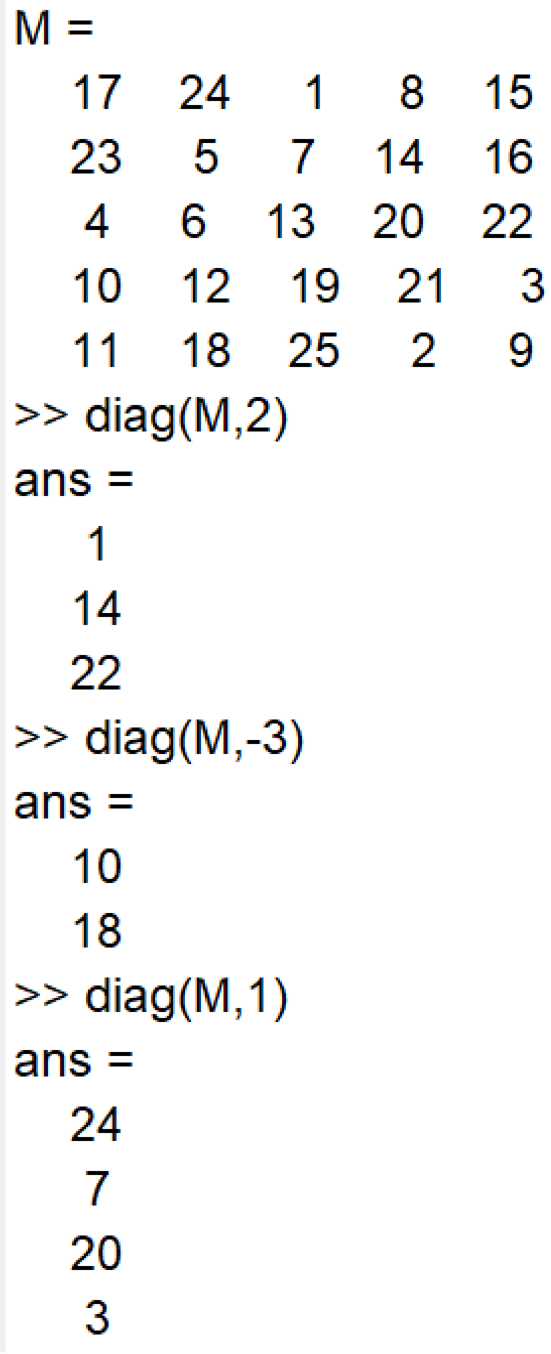


```
V =

    1     2     3     4     5

>> diag(V)

ans =

    1     0     0     0     0
    0     2     0     0     0
    0     0     3     0     0
    0     0     0     4     0
    0     0     0     0     5
```

Extraction of the diagonal vector



```
>> M=ans;
>> diag(M)

ans =

     1
     2
     3
     4
     5
```



```
M =
    17    24     1     8    15
    23     5     7    14    16
     4     6    13    20    22
    10    12    19    21     3
    11    18    25     2     9
>> diag(M,2)
ans =
     1
    14
    22
>> diag(M,-3)
ans =
    10
    18
>> diag(M,1)
ans =
    24
     7
    20
     3
```

Diagonal Matrix

```
if isequal(size(M),[3,3])
    M=[M,M(:,1:2)]
    D(1)=prod(diag(M(:,1:end)))
    D(2)=prod(diag(M(:,2:end)))
    D(3)=prod(diag(M(:,3:end)))
     DD(1)=prod(diag(flipud(M(:,1:end))))
    DD(2)=prod(diag(flipud(M(:,2:end))))
    DD(3)=prod(diag(flipud(M(:,3:end))))
    DET=sum(D)-sum(DD)
end
```

→

```
if isequal(size(M),[3,3])
    M=[M,M(:,1:2)]
    D(1)=prod(diag(M))
    D(2)=prod(diag(M,1))
    D(3)=prod(diag(M,2))
     DD(1)=prod(diag(flipud(M)))
    DD(2)=prod(diag(flipud(M),1))
    DD(3)=prod(diag(flipud(M),2))
    DET=sum(D)-sum(DD)
end
```

Diagonal Matrix

Verify whether a matrix is diagonal

```
>>isequal(M-diag(diag(M)), zeros(size(M)))
```

2	0	5	-1
0	4	0	3
11	-2	2	0
10	0	-3	7

-

2	0	0	0
0	4	0	0
0	0	2	0
0	0	0	7

=

0	0	5	-1
0	0	0	3
11	-2	0	0
10	0	-3	0

Diagonal Matrix

```
Function [trouv]=verify(M)
trouv=false
i=1
while i<= size(M,1) & not (trouv)%trouv==false
    j=1
   while j<=size(M,2) & not(trouv)
       if (i~=j) & M(i,j)~=0
           trouv=true
       end
       j=j+1
   end
   i=i+1
end
```

Special matrices

Triangular matrix

Upper

$$\mathrm{A} \in M_n(K)$$

$$\forall (i, j) \mid i > j, \qquad a_{ij} = 0$$

$$i > j$$

$$\begin{array}{ccc} \mathbf{1 = 1} & \mathbf{1 < 2} & \mathbf{1 < 3} \\ \mathbf{2 > 1} & \mathbf{2 = 2} & \mathbf{2 < 3} \\ \mathbf{3 > 1} & \mathbf{3 > 2} & \mathbf{3 = 3} \end{array}$$

$$\begin{array}{ccc} ? & ? & ? \\ \mathbf{0} & ? & ? \\ \mathbf{0} & \mathbf{0} & ? \end{array}$$

Special matrices

Triangular matrix

Lower

$\mathrm{A} \in M_n(K)$

$\forall (i, j) |\ i < j, \qquad a_{ij} = 0$

$i < j$

$$\begin{array}{ccc} \mathbf{1 = 1} & \mathbf{1 < 2} & \mathbf{1 < 3} \\ \mathbf{2 > 1} & \mathbf{2 = 2} & \mathbf{2 < 3} \\ \mathbf{3 > 1} & \mathbf{3 > 2} & \mathbf{3 = 3} \end{array}$$

$$\begin{matrix} ? & \mathbf{0} & \mathbf{0} \\ ? & ? & \mathbf{0} \\ ? & ? & ? \end{matrix}$$

Special matrices

Triangular matrix

Upper

$\forall(i,j) \mid i > j, \qquad a_{ij} = 0$

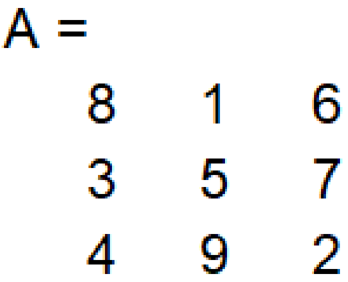
Lower

$\forall(i,j) \mid i < j, \qquad a_{ij} = 0$

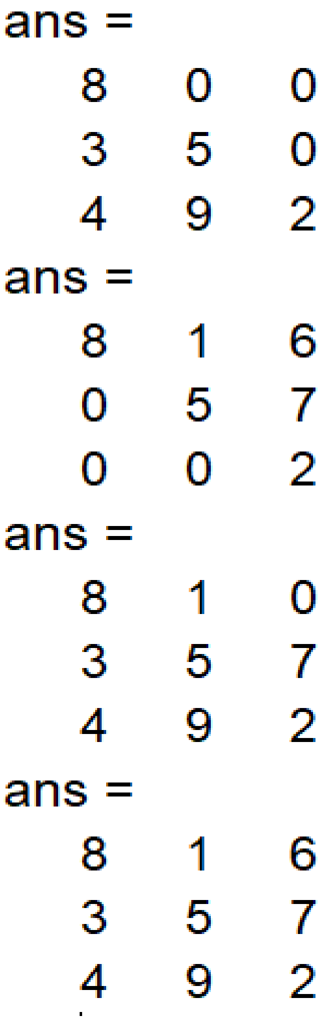
$$\det A = \prod_{i=1..n} a_{ii}$$

Special matrices

Triangular matrix



```
A =
    8     1     6
    3     5     7
    4     9     2
```



tril(A)

```
ans =
    8     0     0
    3     5     0
    4     9     2
```

triu(A)

```
ans =
    8     1     6
    0     5     7
    0     0     2
```

tril(A,1)

```
ans =
    8     1     0
    3     5     7
    4     9     2
```

triu(A,-2)

```
ans =
    8     1     6
    3     5     7
    4     9     2
```

Triangular matrix

Let the matrix M in the workspace,

Extract in A, the upper triangular matrix of M

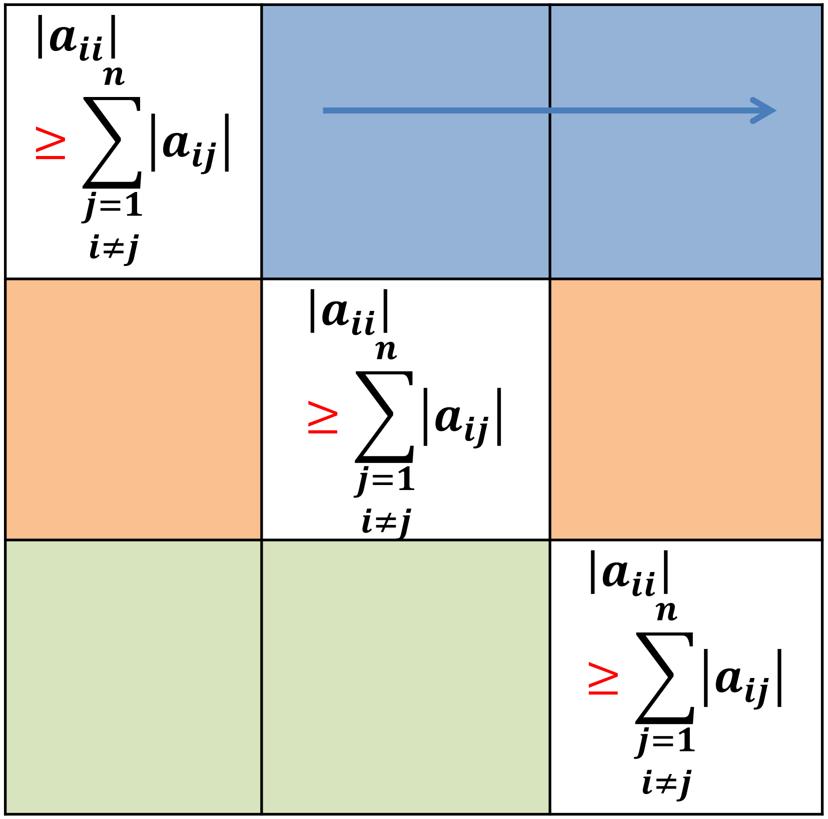
```
for j=1:size(M,2)
    for i=1:j
        A(i,j)=M(i,j)
    end
  end
```

Verify wether A is a upper triangular matrix

```
function[tria_sup]=verif(M)
tria_sup=true
j=1
while (j<=size(M,2)) & (tria_sup)
    i=j+1
    while (i<=size(M,1)) & (tria_sup)
        if (M(i,j))~=0
            tria_sup=false
        end
        i=i+1
    end
    j=j+1
end
end
```

Special matrices

- $A \in M_n(K)$
- **Diagonally dominant matrix (≥)**
- **Strictly diagonally dominant matrix (>)**
- per row



$$|a_{ii}| \geq \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|$$

$$|a_{ii}| \geq \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|$$

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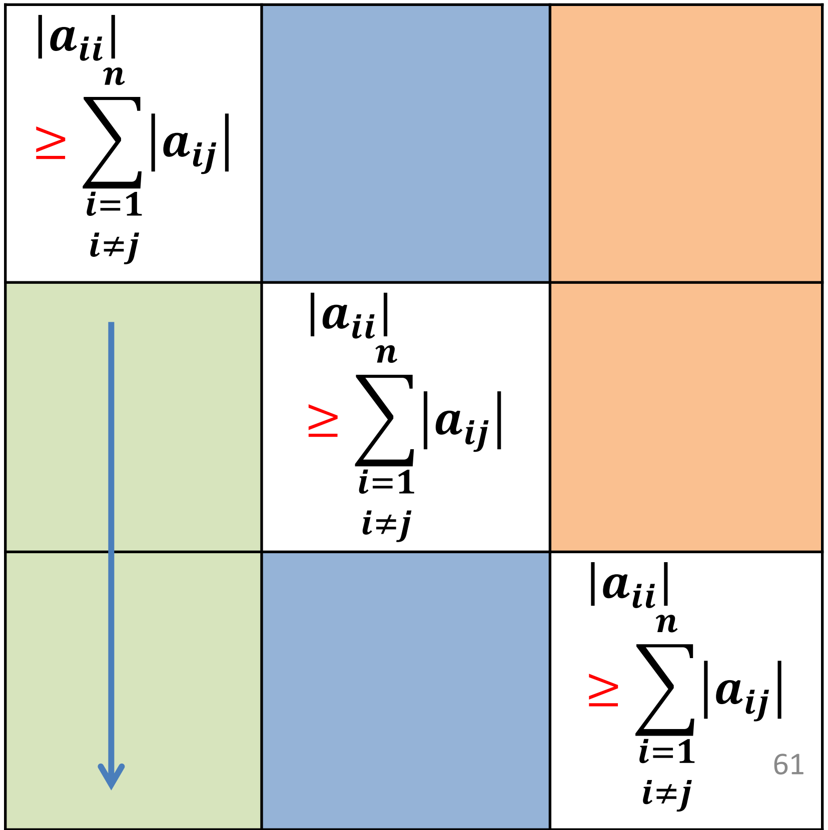
per column



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Special matrices

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & -5 & 2 \\ -1 & 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & -5 & 2 \\ -1 & 2 & 0 \end{pmatrix}$$

Matrix A is diagonally dominant (per row) since

$$\begin{aligned} |4| &\geq |2| + |2| \\ |-5| &\geq |2| + |2| \\ |6| &\geq |-1| + |2| \end{aligned}$$

Matrix B is not diagonally dominant (per row) since

$$|0| < |-1| + |2|$$

A <u>strictly</u> diagonally dominant matrix is always invertible

$$\det A \neq 0$$

Strictly diagonally dominant matrix

```
function[dominan]=verif_domin(M)
dominan=true
i=1
while (i<=size(M,1)) & (dominan)
    s=0
    for j=1:size(M,2)
        if i~=j
           s=  s+abs(M(i,j))
        end
    end
     if abs(M(i,i))<=s
          dominan=false
     end
     i=i+1
end
end
```

Vectorisation

```
A=[5 2 2;2 -5 2;-1 2 6]

d=diag(diag(A))

all(abs(diag(A))>sum(abs(A-d),2))
```

Special matrices

Similar matrices

- $\mathrm{A}, \mathrm{B} \in M_n(K)$
- Two matrices A and B are similar if there exists an invertible matrix P such that :

$$\exists\ P. P^{-1} = I |\ \mathrm{B} = P^{-1} \mathrm{A}\ \mathrm{P}$$

A and B are related by a **similarity transformation**

- $\det(A) = \det(B)$
- $\mathrm{trace}(A) = \mathrm{trace}(B)$

Special matrices

Matrices A=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are similar

Since $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and $P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and $P^{-1}. A. P = B$

Rank of a matrix

The rank of a matrix is the maximum number of linearly independent rows or columns, noted rank(A)

Example :

Let $\mathrm{A} = \begin{matrix} 3 & -4 & -1 \\ 2 & -3 & -1 \end{matrix}$, rank(A) = 2

Calculation for $\boldsymbol{A} \in \boldsymbol{M}_{\boldsymbol{n},\boldsymbol{m}}$:

- $\mathrm{A} = 0_{\mathrm{n,m}} \Leftrightarrow \mathrm{rank(A)} = 0$
- $if\ C_k = \sum_{\substack{i=1 \\ i \neq k}}^{\min(n,m)} \alpha_i \cdot C_i$ then $\mathrm{rank}(A) = \mathrm{rank}(A) - 1$
- $\mathrm{rank(A)} \leq \min(\mathrm{n,m})$

Rank of a matrix

Example 2 :

$$\mathrm{A} = \begin{pmatrix} 1 & 2 & -\frac{1}{2} & 0 \\ 2 & 4 & -1 & 0 \end{pmatrix} \in M_{2,4}(K)$$

$$\text{rank}(A) = \text{rank}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -\dfrac{1}{2} \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\text{rank}(\mathrm{A}) = 1$$

Synthesis of Concepts

For $A \in M_n(\mathbb{K})$

$$A\ is\ invertible \Leftrightarrow$$

$$det(A) \neq 0 \Leftrightarrow$$

$$\mathrm{rank}(A) = n \ \Leftrightarrow$$

$$\nexists\ C_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i \cdot C_i \Leftrightarrow$$

$$\nexists\ L_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i \cdot L_i \Leftrightarrow$$